Discussion Paper

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no. 86 March, 1993

Sustainability when Resource Management has Stochastic Consequences

by

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Abstract

Sustainability is usually defined as a requirement to each generation to manage the resource base such that the average quality of life that it ensures itself can be shared by all future generations. When the consequences of each generation's management of the resource base are not deterministic, this definition can, however, not be used. The purpose of this note is to present a characterization of the usual definition of sustainability that can be extended to, and hence provides a definition in, the case with risk. The properties of this proposed definition in the case with risk are investigated.

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The idea of this paper arose through discussions with Eirik Amundsen, Erling Moxnes, and Bjørn Sandvik. In particular, we would like to thank the latter as well as Trond Olsen and seminar participants at the EEAC in Dublin, for helpful comments.

Asheim has received financial support from the Norwegian Ministry of the Environment and the Norwegian Research Council for Applied Social Sciences through the Norwegian Research Centre in Organization and Management.

1 Introduction

The notion of 'sustainable development' was introduced on the political agenda by the World Commission on Environment and Development through its report (WCED, 1987), also called the Brundtland Report. The Brundtland Report does not give a precise definition of the notion of 'sustainable development'. The quotation that is usually taken as a point of departure is the following: "Sustainable development is a development that meets the needs of the present without comprising the ability of future generations to meet their own needs" (WCED, 1987, p. 43) The Brundtland Report looks at sustainability both as a requirement for <u>intra</u>generational justice and as a requirement for <u>inter</u>generational justice. If we here choose to limit the discussion by considering sustainability to be a requirement for intergenerational justice, sustainability requires from our generation not to use more than our fair share of the resource base. More, precisely, sustainability is defined as a *requirement to our generation to manage the resource base such that the average quality of life that we ensure ourselves can potentially be shared by all future generations.*[1]

The notion 'quality of life' is meant to include everything that influences the situation in which people live. Hence, the notion includes much more than material consumption. It is intended to capture the importance of health, culture, and nature. One limitation is of importance, though: The quality of life does not include the welfare that people derive from their children's consumption. Likewise, only the instrumental value in nature (i.e., recognized value to humans) is included in the quality of life, not the intrinsic value in nature (i.e., value in its own right regardless of human experience).[2] The rationale behind these limitations is that it is desirable to separate the definition of sustainability from the forces that can motivate our

[1]This is the definition of sustainability that was suggested by NAVF (1990, p. 233).

[2]See Pearce & Turner (1990, pp. 12-15) for an elaboration on the difference between instrumental and intrinsic value.

generation to act in accordance with the requirement of sustainability.

It is possible that our generation is about to use the resource base so as to ensure ourselves a quality of life that cannot be shared by all future generations. In such a case sustainability requires that we today reduce the exploitation of the resource base.

If the requirement of sustainability as defined above is not extended to later generations, it does not rule out that some later generation uses the resource base to ensure itself an average quality of life that cannot be shared by its successors. It seems, however, odd not to let sustainability be a requirement to later generations as well. In particular, it would be unreasonable for our generation to have the welfare of distant generations in mind if we believed that the intermediate generations would not take part in an effort to give these generations their fair share of the resource base. Extending the requirement of sustainability to later generations yields the following definition of a sustainable development:

> A development is sustainable if it involves a non-decreasing average quality of life.

Furthermore, it places the following requirement on our generation:

> Our generation's management of the resource base is sustainable if it constitutes the first part of a feasible sustainable development.

This is the interpretation of sustainability which has been suggested in a number of references.[3] Sustainability in the above sense is a natural requirement of intergenerational justice because it can be shown under given conditions that if development is not sustainable there exists another development that increases the total sum to

[3]The idea of defining sustainability in this way dates at least back to Tietenberg (1984) and seems at this point in time to be fairly widely accepted; see, e.g. Repetto (1986), Pezzey (1989), Mäler (1989), and Amundsen et al. (1991). A critical assessment of this interpretation of sustainability is given by Pearce et al. (1989, pp. 32 & 49).

be shared between the generations and, in addition, shares it in a more egalitarian way (Asheim, 1991).

If a notion of sustainability is to be of practical importance in the real management of natural and environmental resources, it is essential that the notion is operational. The notion becomes operational if the following question can be answered: What kind of rules must our generation follow in order to manage the resource base in such a way that it constitutes a first part of a sustainable development? The problem of finding such rules can only – if at all – be resolved through an analysis of the long-term global production possibilities.

Even if such an analysis were feasible, there remains a major deficiency with the suggested definition of sustainability. Above, sustainability has been defined without taking account of risk and uncertainty. This is unsatisfactory since the long-term consequences of human activity are not deterministic; in particular, risk and uncertainty are present in the management of natural and environmental resources. The assumption of full certainty is inappropriate, hence, for studying the most important issues related to sustainability. We do not know for sure whether the future quality of life will be increasing or decreasing. The crucial question is whether the risk of decreasing future quality of life is acceptable. To raise and study this question, the definition of sustainability must be extended to the case where the resource management does not have deterministic consequences, and methods for solving for sustainable resource management policies must be developed. This is the purpose of the present note.

This note suggests to extend the applicability of the definition of sustainability in the following manner: First, in Section 2, it is shown that the definition of sustainability given above can under given conditions be rewritten as follows:

> Sustainability is a requirement to each generation to manage the resource base in such a way that its average quality of life can be shared by the next generation even if the latter abide by the requirement of sustainability.

Note that this characterization of sustainability — as opposed to the earlier definition — does not compare the quality of life of the current generation with the quality of life of *all* future generations. Rather, it compares the quality of life of the current generation with the quality of life of only the next generation under the proviso that the latter generation acts in accordance with sustainability. This is the reason why this characterization can, as shown in Section 3, be generalized to the case where the consequences of each generation's resource management are not deterministic:

> Sustainability is a requirement to each generation to manage the resource base in such a way that its quality of life can be shared by the next generation in the sense of certainty equivalents even if the latter abide by the requirements of sustainability.

By 'its quality of life can be shared in the sense of certainty equivalents', we mean that the certainty equivalent of the next generations quality of life is not exceeded by the current generations quality of life. Note that this definition only requires that certainty equivalents are well defined. The independence axiom of expected utility is not needed, hence this approach allows for evaluations of risk where negative catastrophic events with small probabilities are given a higher weight than expected utility allows.

The note is concluded in Section 4 by two examples that aim at illustrating the concept of sustainability when resource management does not have deterministic consequences.

Some of the concepts, notation, and analysis are influenced by the work of Greenberg (1990). This note is not, however, an application of his 'theory of social situations'.

2 Sustainability without risk

The purpose of the present section is to give a characterization of the definition of sustainability that was informally presented in the introduction for the case without risk. This characterization can then be used in Section 3 to define sustainability for the case in which the consequences of each generation's management of the resource base are not deterministic.

For this purpose, we introduce the following formalism. Assume that there is an infinite number of generations 1, 2, .. that do not overlap. Hence, the society is potentially infinitely lived. Denote by y_t ($\in Y_t$) the position that generation t inherits. It is natural to think of y_t as a vector of the stocks that makes up the resource base. Interpret c_t ($\in \Re_+$) – the average consumption of generation t – as a scalar that indicates its average quality of life. The analysis allows for a population that varies exogenously over time since c_t denotes the consumption of a member of generation t under the assumption that the total consumption of generation t is distributed evenly among its members. Issues relating to redistribution within each generation are, however, not touched upon. In particular, the responsibility for intragenerational distribution is left to the generation in question, assuming that its decisions with respect to redistribution among its own members do not affect the possibility for redistribution between generations.[4]

As mentioned in the introduction, c_t comprises much more than material consumption, but does not include the welfare that generation t may derive from later generations' consumption; neither does it include the welfare that generation t may derive from the existence and preservation of nature regardless of human experience. Such altruism for future generations and intrinsic value in nature are excluded, since there is an argument to be made in favor of distinguishing the definition of sustain-

[4]The validity of this assumption is easy to question. In particular, does the unequal distribution of wealth within our own generation prevent us from taking proper care of the resource base? Still, such problems are outside the scope of this note.

ability from the forces that are instrumental in attaining it.

The resource management of generation t is described by a pair (c_t, y_{t+1}) where c_t is the consumption level of generation t and y_{t+1} is the bequest to generation $t+1$. Let $\mathcal{T}_t(y_t)$ be the transformation set at time t in the position y_t meaning that, if generation t inherits y_t, the resource management pair (c_t, y_{t+1}) is feasible if and only if $(c_t, y_{t+1}) \in \mathcal{T}_t(y_t)$. Note that generation t is here assumed to be able to choose in a deterministic way the resource base that generation $t+1$ inherits. Technological progress is allowed to enter exogenously through the time dependency of the transformation sets $\mathcal{T}_t()$ for $t \geq 1$, and endogenously through the position y_t if some elements of this vector include accumulated knowledge resulting from learning and research activities.

Sustainability can now formally be defined for the case without risk. Write ${}_t\mathbf{c} = (c_t, c_{t+1}, ...)$ (and correspondingly for other sequences), and let $\underline{c}(\geq 0)$ denote the subsistence level.

Definition 1 *The resource management pair (c_t, y_{t+1}) is sustainable at time t in the position y_t if and only if $c_t \geq \underline{c}$ and there exist ${}_{t+1}\mathbf{c}$ and ${}_{t+2}\mathbf{y}$ such that ${}_t\mathbf{c}$ is non-decreasing and, for each $s \geq t$, $(c_s, y_{s+1}) \in \mathcal{T}_s(y_s)$. The consumption level c_t is sustainable at time t in the position y_t if and only if it is part of a sustainable pair (c_t, y_{t+1}).*

Hence, generation t's consumption level is sustainable if it is above the subsistence level and constitutes the first part of a feasible program with non-decreasing consumption.

The requirement of sustainability does not prescribe an optimal consumption level for generation t. Rather, it gives a set of allowable consumption level and bequest pairs. In general sustainability prescribes to generations 1,2,3,... a *standard of behavior* (SB) ${}_1\sigma = (\sigma_1, ...)$, where $\sigma_t(y_t)$ is the set of allowable resource management pairs (c_t, y_{t+1}) at time t in the position y_t. What kind of SB is equivalent to the requirement of sustainability? In this note we will require the three following properties:

- A SB can only prescribe a feasible action.
- Furthermore, we require that a consumption level can be allowed by a SB only if it does not fall below the subsistence level.
- Finally, it must be required of a SB that if the consumption level of generation t is allowable, then generation $t+1$ must be able to ensure itself an allowable consumption level that is at least as high.

These requirements are expressed in the notion of non-decreasing SB, defined as follows.

Definition 2 *A SB* ${}_1\sigma$ *is* non-decreasing *if for all* $t \geq 1$ *and* $y_t \in Y_t$, *(i)* $\sigma_t(y_t) \subseteq \mathcal{T}_t(y_t)$, *and (ii)* $(c_t, y_{t+1}) \in \sigma_t(y_t)$ *implies that* $c_t \geq \underline{c}$, *and that there exists* $(c_{t+1}, y_{t+2}) \in \sigma_{t+1}(y_{t+1})$ *such that* $c_{t+1} \geq c_t$.

The following lemma is a straightforward implication of this definition:

Lemma 1 *If* $(c_t, y_{y+1}) \in \sigma_t(y_t)$ *for a non-decreasing SB,* ${}_1\sigma$*, then there exists* ${}_{t+1}\mathbf{c}$ *and* ${}_{t+2}\mathbf{y}$ *such that* ${}_t\mathbf{c}$ *is non-decreasing and, for each* $s \geq t$, $(c_s, y_{s+1}) \in \sigma_s(y_s)$.

Proof The result is obtained through repetitive use of the definition of a non-decreasing SB. □

Hence, if a non-decreasing SB allows that generation t ensures itself a particular consumption level, then its consumption level constitutes the first part of a feasible program with non-decreasing allowable consumption

Being a non-decrasing SB is not, however, sufficient to characterize sustainability; in particular, ${}_1\sigma^{\emptyset}$ – with $\sigma_t^{\emptyset}(y_t) = \emptyset$ for all $t \geq 1$ and $y_t \in Y_t$ – is non-decreasing. Therefore, consider the notion of a maximal non-decreasing SB, where maximality is defined w.r.t. set inclusion.

Lemma 2 *Let ${}_1\sigma^*$ be defined by $(c_t, y_{t+1}) \in \sigma_t^*(y_t)$ if and only if $c_t \geq \underline{c}$ and there exists ${}_{t+1}\mathbf{c}$ and ${}_{t+2}\mathbf{y}$ such that ${}_t\mathbf{c}$ is non-decreasing and for all $s \geq t$, $(c_s, y_{s+1}) \in \mathcal{T}_s(y_s)$. Then ${}_1\sigma^*$ is the unique maximal non-decreasing SB.*

Proof: Let $(c_t, y_{t+1}) \in \sigma_t^*(y_t)$. By the definition of ${}_1\sigma^*$ we can choose ${}_{t+1}\mathbf{c}$ and ${}_{t+2}\mathbf{y}$ such that ${}_t\mathbf{c}$ is non-decreasing and for each $s \geq t$, $(c_s, y_{s+1}) \in \mathcal{T}_s(y_s)$. Then $c_{t+1} \geq c_t \geq \underline{c}$ and ${}_{t+2}\mathbf{c}$ and ${}_{t+3}\mathbf{y}$ satisfies that ${}_{t+1}\mathbf{c}$ is non-decreasing and, for all $s \geq t+1$, $(c_s, y_{s+1}) \in \mathcal{T}_s(y_s)$; i.e., $(c_{t+1}, y_{t+2}) \in \sigma_{t+1}^*(y_{t+1})$. Hence, there exists $(c_{t+1}, y_{t+2}) \in \sigma_{t+1}^*(y_{t+1})$ such that $c_{t+1} \geq c_t$, which implies that ${}_1\sigma^*$ is a non-decreasing SB. By Lemma 1, ${}_1\sigma^*$ is the unique maximal non-decreasing SB. □

Hence, by Lemma 2 there exists a unique non-decreasing SB which allows a resource management pair if and only if it is sustainable according to Definition 1. Lemma 2 therefore yields the following characterization of sustainability when the consequences of each generation's management of the resource base are deterministic.

Theorem 1 *A resource management pair (c_t, y_{t+1}) is sustainable at time t in position y_t if and only if $(c_t, y_{t+1}) \in \sigma_t^*(y_t)$, where ${}_1\sigma^*$ is the unique maximal non-decreasing SB.*

Note that ${}_1\sigma^*$ may be empty-valued. If it is impossible at a position y_t to find a ${}_t\mathbf{c}$ and ${}_{t+1}\mathbf{y}$ such that $c_t \geq \underline{c}$, ${}_t\mathbf{c}$ is non-decreasing and $(c_s, y_{s+1}) \in \mathcal{T}_s(y_s)$ for all $s \geq t$, then we must have $\sigma^*(y_t) = \emptyset$.

3 Sustainability with risk

In the situation with risk, each generation t cannot in a deterministic way choose the position y_{t+1} that the following generation will inherit; rather, it can choose a stochastic variable with values in Y_t. To be more precise, let $(\Omega, \mathcal{F}, Q)$ be a countable[5] measure space, and $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots \subset \mathcal{F}$, an increasing sequence of

[5]The countability assumption is introduced to avoid measurability problems.

σ–algebras, where $\mathcal{F}_t$ represents the information available at time t. The resource management of generation t is described by a pair $(c_t, \tilde{y}_{t+1})$ where c_t is the consumption level of generation t and $\tilde{y}_{t+1}$ is the $\mathcal{F}_{t+1}$-measurable stochastic bequest to generation $t+1$. Let $\mathcal{T}_t(y_t)$ be the transformation set at time t in the position y_t, meaning that if generation t inherits y_t, the resource management pair $(c_t, \tilde{y}_{t+1})$ is feasible if and only if $(c_t, \tilde{y}_{t+1}) \in \mathcal{T}_t(y_t)$.

This formulation is in accordance with the way that problems of stochastic control are usually put. In particular, the control variables of generation t are, in addition to the consumption level c_t, variables that allows for a choice of random variables with different probability distribution for the bequest to generation $t+1$. Generation t may e.g., for a given choice of c_t be able to choose between a safe and a risky technique. Such choices are illustrated by the first example of section 4.

The generality of this formulation can be illustrated by noting that it allows for irreversible decisions that expose all future generations to risk. Production of environmentally dangerous waste that is active infinitely long and, hence, poses a threat to all future generations is one such example. This is captured in the present framework by:

- letting the i'th element of y_t, $y_{i,t}$ represent the stock of this waste at time t,
- ruling out that generation t can make decisions such that $y_{i,t+1} < y_{i,t}$ with positive probability
- assuming the elements of $\tilde{y}_{t+1}$ corresponding to environmental quality are more volatile the greater that $y_{i,t}$ is.

In order to apply the characterization result of Theorem 1 to define sustainability when the consequences of resource management are not deterministic, it is necessary to define what is meant by a non-decreasing SB in the case with risk.

Let $P_t : Y_t \mapsto \Re_+ \times \mathcal{M}(Y_{t+1})$ where $\mathcal{M}(Y_t)$ is the set of possible random variables $\tilde{y}_t : \Omega \mapsto Y_t$. P_t is a function that for each possible position defines a particular resource management pair. The function P_t will be referred to as a resource management policy. We let $\mathcal{P}_t$ denote the set of feasible resource management policies at time t.

Note that the pair $(P_t, \tilde{y}_t)$ defines a random variable $\tilde{c}_t(\omega) = P_{c,t}(\tilde{y}_t(\omega))$; the quality of life of generation t. The idea is now that a resource management pair is sustainable at time t in position y_t if $\tilde{c}_{t+1}$ is 'no worse than' c_t even if generation $t+1$ acts according to a sustainable resource management pair. The presence of uncertainty poses additional problems to the interpretation of 'no worse than'[6]. According to who's preferences is $\tilde{c}_{t+1}$ no worse than c_t? We may imagine that the two generations t and $t+1$ meets behind a 'veil of ignorance' (like in Rawls (1972)). Suppose all rational individuals, not knowing which generation they will be born into, will agree on a common preference ordering over random variables describing quality of life, then $\tilde{c}_{t+1}$ is no worse than c_t if this is so in the common preference ordering.

In the case of unanimity behind a veil of ignorance the preferences are well defined. It may be argued that the preferences should be representable by expected utility. Expected utility is, however, not consistent with observed behaviour under uncertainty. Of especial importance in this context is the "fanning out" property of behaviour, (see Machina (1982)), which implies that events with huge negative consequences but with small probability is given extra weight. Also the normative status of expected utility as a theory for individual choices under risk has become controversial during the last decades. See e.g. Fishburn (1988) for a discussion. Moreover, expected utility only focuses on the individuals utility and is thus subject to Sen's (1979) criticism of 'welfarism'.Without unanimity behind the veil of ignorance there is a problem of aggregating the preferences. As shown in Arrow's

[6]There are of course problems in defining 'no worse than' in the deterministic case. It is far from obvious how quality of life should be measured, and the choice of measure is also a choice of interpretation of 'no worse than' in the deterministic case. The problem of defining quality of life is important, but not a central issue to this paper though. We have thus disregarded the problem

(1951) seminal impossibility theorem, there are logical problems with aggregating preferences.

There are some additional problems with the idea of a 'veil of ignorance' in this context. In particular, the generation $t+1$ may not correspond to the same individuals in all states of the world $\omega \in \Omega$. However, we will here retain the assumption made in section 2 that the population varies exogenously with time, and that it does not depend upon the state of the world.

The discussion about the interpretation of $\tilde{c}_{t+1}$ being 'no worse than' c_t is beyond the scope of the present paper. Different points of view on the problems indicated above may give rise to different types of preferences over random variables. To allow for studies of different preferences within the framework presented in this paper, we will base the following analysis on as weak assumptions as possible. At this stage we will assume that for each random variable $\tilde{c}_{t+1}$ *there exists a certainty equivalent*, $\mu(\tilde{c}_{t+1} \mid \mathcal{F}_s)$, given the information $\mathcal{F}_s$ available for generation s. For simplicity we will write $\mu(\tilde{c}_{t+1})$ for $\mu(\tilde{c}_{t+1} \mid \mathcal{F}_t)$.

Now we will proceed to show that with the existence of certainty equivalents, the notion of non-decreasing SB can be extended to the case with risk.

For a given SB, and random inheritance $\tilde{y}_t$, let $\bar{\mu}(\sigma_t(\tilde{y}_t))$ denote the set of quality of life dominated by a ceratinty equivalent consistent with σ_t. Formally,

$$c \in \bar{\mu}(\sigma_t(\tilde{y}_t)) \tag{1}$$

if there exists a policy P_t with $P_t(y) \in \sigma_t(y)$ for all $y \in supp(\tilde{y}_t)$ [7] and such that $c \leq \mu(P_{c,t}(\tilde{y}_t))$. Consider the operator T_t defined by

$$T_t(\sigma_{t+1})(y_t) = \{(c_t, \tilde{y}_{t+1}) \in T_t(y_t) : \underline{c} \leq c_t \in \bar{\mu}(\sigma_{t+1}(\tilde{y}_{t+1}))\} \tag{2}$$

Note that T_t is isotonic, i.e. if $\sigma_{t+1}(y) \subseteq \sigma'_{t+1}(y)$, for all $y \in Y_{t+1}$, then $T_t(\sigma_{t+1})(y_t) \subseteq T_t(\sigma'_{t+1})(y_t)$ for all $y_t \in Y_t$. This property is essential to the proofs of

[7] In the following the 'support' denotes the set of all points with positive probability, not the closure of this set. Thus supp$(\tilde{y}) = \{y : \text{Prob}\{\tilde{y} = y\} > 0\}$

Theorem 2 and 3.

Definition 3 *A SB* ${}_1\sigma$ *is* non-decreasing *if for all* $t \geq 1$ *and all* $y_t \in Y_t$, $\sigma_t(y_t) \subseteq T_t(\sigma_{t+1})(y_t)$

Combining the inclusion $\sigma_t(y_t) \subseteq T_t(\sigma_{t+1})(y_t)$ and the definition of the operator T_t, we note that the requirement to a non-decreasing SB is as without risk.

- The policy has to be feasible, since $T_t(\sigma_{t+1})(y_t) \subseteq \mathcal{T}_t(y_t)$.
- A consumption level can be allowed by a non-decreasing SB only if it is not below subsistence level, that is $(c_t, \tilde{y}_{t+1}) \in \sigma_t(y_t)$ implies $c_t \geq \underline{c}$.
- Finally, if the consumption level of generation t is allowable by a non-decreasing SB, then the generation $t+1$ must be able to ensure itself a probability distribution of consumption that is 'no worse than' the certain consumption level of generation t.

This definition of non-decreasing is local, since only consumption at t and $t+1$ is compared. What about relative preferability of the consumption at time t and s, where $s - t \geq 2$? Will a person who, for all $s \geq t$ weakly prefers $\tilde{c}_{s+1}$ to $\tilde{c}_s$ *given the information* $\mathcal{F}_s$, also prefer $\tilde{c}_s$ to c_t given the information $\mathcal{F}_t$? Since the information differs at each stage, we cannot simply apply an assumption on transitivity. To study this question we make the following definition.

Definition 4 *The certainty equivalent has extendable monotonicity if for all stochastic processes* ${}_1\tilde{\mathbf{c}}$ *with the property* $\mu(\tilde{c}_{t+1}) \geq c_t$ *for all* $t \geq 1$, *then we have* $\mu(\tilde{c}_s \mid \mathcal{F}_t) \geq c_t$, *for all* $s > t \geq 1$.

The following lemma is straightforward

Lemma 3 *If the preferences satisfy expected utility then the certainty equivalent has extendable monotonicity.*

Proof: When preferences satisfy expected utility, $\mu(\tilde{c}_s \mid \mathcal{F}_t) \geq c_t$ is equivalent to $E[u(\tilde{c}_s) \mid \mathcal{F}_t] \geq u(c)$. Suppose the $\mu(\tilde{c}_s \mid \mathcal{F}_t) \geq c_t$ holds for $s - t \leq k$. According to the premisses of extendable monotonicity, this holds for $k = 1$. Now chose $s = t + k + 1$ Then

$$\begin{aligned} E[u(\tilde{c}_s) \mid \mathcal{F}_t] &= E[E[u(\tilde{c}_s) \mid \mathcal{F}_{s-1}] \mid \mathcal{F}_t] && (3) \\ &\geq E[u(\tilde{c}_{s-1}) \mid \mathcal{F}_t] && (4) \\ &\geq u(c_t) && (5) \end{aligned}$$

This proves the lemma by induction. □

To make a natural extension of the results of section 2, we will relate the definition of sustainability to the concept of a *maximal* non-decreasing SB. The following theorem establishes the existsence of a maximal non-decreasing SB. The proofs of theorem 2 and 3 are inspired by Tarskys (1955) argument for the existence of fixed points for isotonic correspondences.

Theorem 2 *There exists a unique maximal non-decreasing SB.*

Proof: Let $\mathcal{A}$ be the set of non-decreasing SBs. We denote a typical element as ${}_1\sigma^\alpha$ whith $\alpha \in \mathcal{A}$. Note that ${}_1\sigma^\emptyset$, defined by $\sigma_t^\emptyset(y_t) = \emptyset$ for all t and all y_t is non-decreasing since $T_t(\sigma_{t+1}^\emptyset)(y_t) = \emptyset$. Hence the set $\mathcal{A}$ of non-decreasing SBs is non-empty. Define a SB ${}_1\sigma^*$ by, for all $t \geq 1$ and $y_t \in Y_t$,

$$\sigma_t^*(y_t) = \bigcup_{\alpha \in \mathcal{A}} \sigma_t^\alpha(y_t)$$

Note that ${}_1\sigma^*$ is non-decreasing since for any $\alpha \in \mathcal{A}$, $\sigma_t^\alpha(y_t) \subseteq T_t(\sigma_{t+1}^\alpha)(y_t) \subseteq T_t(\sigma_{t+1}^*)(y_t)$ for all t and y_t. Hence, ${}_1\sigma^*$ is the unique maximal non-decreasing SB. □

Definition 5 *A resource management pair $(c_t, \tilde{y}_{t+1})$ is sustainable at time t in the position y_t if and only if $(c_t, \tilde{y}_{t+1}) \in \sigma_t^*(y_t)$ where ${}_1\sigma^*$ is the unique maximal non-decreasing SB.*

The consumption level c_t is sustainable at time t in the position y_t if and only if it is part of a sustainable resource management pair $(c_t, \tilde{y}_{t+1})$.

Theorem 3 *The unique maximal SB ${}_1\sigma^*$ satisfies $\sigma_t^*(y_t) = T_t(\sigma_{t+1}^*)(y_t)$ for all $t \geq 1$ and $y_t \in Y_t$.*

Proof: Suppose there exists t and y_t such that $\sigma_t^*(y_t) \subset T_t(\sigma_{t+1}^*)(y_t)$. Construct ${}_1\sigma$ by

$$\sigma_t(y_t) = T_t(\sigma_{t+1}^*)(y_t) \tag{6}$$

and $\sigma_s(y_s) = \sigma_s^*(y_s)$ otherwise. Note that ${}_1\sigma$ is non-decreasing since $\sigma_s(y_s) \subseteq T_s(\sigma_s)(y_s)$ for all s and y_s. Hence, by the construction of ${}_1\sigma^*$, $\sigma_s(y_s) \subseteq \sigma_s^*(y_s)$ for all s and y_s. A contradiction is established since $\sigma_t(y_t) \not\subseteq \sigma_t^*(y_t)$. □

Let the 'recursive support', $S_{s,t}(y_t; {}_1\mathbf{P})$, starting from y_t using ${}_1\mathbf{P}$ be the set of states y_s that can be reached at time s when starting in y_t and using the policy ${}_1\mathbf{P}$. Usually, when there can be no misunderstanding, $S_{s,t}(y_t; {}_1\mathbf{P})$ is simply written $S_{s,t}(y_t)$. S is defined recursively as $S_{t,t}(y_t) = \{y_t\}$, and for $s > t$,

$$S_{s,t}(y_t) = \left\{ y \in Y_s : \begin{array}{c} y \in \text{supp}(\tilde{y}_s), \text{ with } (c_{s-1}, \tilde{y}_s) = P_{s-1}(y_{s-1}) \\ \text{for some } y_{s-1} \in S_{s-1,t}(y_t) \end{array} \right\} \tag{7}$$

Lemma 4 *Let ${}_1\mathbf{P} \in_1 \mathcal{P}$ be the resource management policies of generation 1,2,... If $P_t(y_t) = (c_t, \tilde{y}_{t+1}) \notin \sigma_t^*(y_t)$ for some t and y_t, then there exists $s \geq t$ and $y_s \in S_{s,t}(y_t)$, such that $P_{c,s}(y_s) < \underline{c}$ or $P_{c,s}(y_s) > \mu(P_{c,s+1}(P_{y,s}(y_s)))$.*

Proof Consider the SB ${}_1\sigma$ defined by

$$\sigma_s(y_s) = \begin{cases} \{P_s(y_s)\} & \text{for all } y_s \in S_{s,t}(y_t) \\ \emptyset & \text{otherwise} \end{cases} \tag{8}$$

for all $s \geq t$. By the construction of ${}_1\sigma^*$, ${}_1\sigma$ is not non-decreasing. Hence there exists s and y_s such that $\sigma_s(y_s) \not\subseteq T_s(\sigma_{s+1})(y_s)$. Since $\sigma_s(y_s) \neq \emptyset$, it follows that $s \geq t$ and $y_s \in S_{s,t}(y_t)$ □

In order to investigate the normative properties of Definition 5, consider the following assumption.

Assumption 1 (Costless storage) *If* $y_t \in Y_t$, $P_t \in \mathcal{P}_t$, $P_{t+1} \in \mathcal{P}_{t+1}$ *satisfies* $P_{c,t}(y_t) > \mu(P_{c,t+1}(P_{y,t}(y_t))$ *then for each* $\Delta c \in (0, P_{c,t}(y_t) - \mu(P_{c,t+1}(P_{y,t}(y_t)))$ *there exists* $P'_t \in \mathcal{P}_t$ *and* $P'_{t+1} \in \mathcal{P}_{t+1}$ *such that*

$$P'_{c,t}(y_t) = P_{ct}(y_t) - \Delta c,$$

$$\mu(P'_{c,t+1}(P'_{y,t}(y_t))) = \mu(P_{c,t+1}(P_{y,t}(y_t))) + \Delta c,$$

and

$$P'_{y,t+1}(P'_{y,t}(y_t))) = P_{y,t+1}(P_{y,t}(y_t))).$$

According to Dalton's (1920) *principle of transfers*, the transfer from a richer to a poorer individual is desirable. Here the individuals correspond to generations, hence the following constitutes a necessary condition for the fulfillment of Dalton's principle of transfers. The policy ${}_1\mathbf{P} \in_1 \boldsymbol{\mathcal{P}}$ satisfies Dalton's (1920) principle of transfers given t and y_t only if there do not exist $s \geq t$ and $y_s \in S_{s,t}(y_s)$ and ${}_1\mathbf{P}' \in_1 \boldsymbol{\mathcal{P}}$ such that,

$$P_{c,s}(y_s) > \mu(P_{c,s+1}(P_{y,s}(y_s))$$

$$P'_{c,s}(y_s) = P_{c,s}(y_s) - \Delta c,$$

and

$$\mu(P'_{c,s+1}(P'_{y,s}(y_s))) = \mu(P_{c,s+1}(P_{y,s}(y_s))) + \Delta c,$$

with $\Delta c \in (0, P_{c,s}(y_s) - \mu(P_{c,s+1}(P_{y,s}(y_s)))$, and

$$P'_{y,s+1}(P'_{y,s}(y_s))) = P_{y,s+1}(P_{y,s}(y_s))).$$

By adopting assumption 1, the following normative foundation for the definition of sustainability can be laid.

Theorem 4 *Let* ${}_1\mathbf{P} \in_1 \boldsymbol{\mathcal{P}}$ *be the resource management policies of generations 1,2,... If, for some* t *and* $y_t \in Y_t$, *the resource management pair* $P_t(y_t) = (c_t, \tilde{y}_{t+1})$ *is not sustainable, then either*

1. with positive probability $c_s < \underline{c}$ at some $s \geq t$ or,

2. ${}_1\mathbf{P}$ does not satisfy Dalton's (1920) principle of transfer given t and y_t.

Proof: The result follows directly from Lemma 4, the definition of sustainability, the assumption of costless storage and the necessary condition for the fulfillment of Dalton's principle. □

In the following we will demonstrate that the maximal non-decreasing SB can be studied with tools from dynamic programming. The discussion and notation is based on Bertsekas and Shreve (1978).

Let $\mathcal{S}_t = \{y_t \in Y_t : \sigma_t^*(y_t) \neq \emptyset\}$ denote the set of states where a sustainable resource management pair exists. Let ${}_1\mathbf{J}^*$ denote the upper bond on sustainable consumption. Formally, for all t and all $y_t \in \mathcal{S}_t$,

$$J_t^*(y_t) = \sup\{c_t : \exists \tilde{y} \text{ such that } (c_t, \tilde{y}) \in \sigma^*(y_t)\} \tag{9}$$

In the following all statements about J–functions are implicitly assumed to be restricted to the set $\mathcal{S}_t$. We will demonstrate that J^* can be directly characterized by an operator equation. Let the operator $\hat{T}_t$ be defined as follows:

$$\hat{T}_t(J)(y_t) = \sup\left\{ c_t : \begin{array}{l} \exists \tilde{y} \text{ such that } supp(\tilde{y}) \subseteq \mathcal{S}_t, \\ (c_t, \tilde{y}) \in \mathcal{T}_t(y_t) \text{ and } \underline{c} \leq c_t \leq \mu(J(\tilde{y})) \end{array} \right\} \tag{10}$$

if this set is nonempty, and $\hat{T}_t(J)(y_t) = -\infty$ otherwise. Furthermore, ${}_1\mathbf{P}$ is said to be a sustainable policy if and only if for all t and all $y_t \in \mathcal{S}_t$, $P_t(y_t) \in \sigma_t^*(y_t)$ and $P_{c,t}(y_t) \leq \mu(P_{c,t+1}(P_{y,t}(y_t)))$. Note that it follows from lemma 4 that c_t is sustainable at time t in the position y_t if and only if there exists a sustainable policy ${}_1\mathbf{P}$ such that $c_t = P_{c,t}(y_t)$.

Lemma 5 *Let ${}_1\mathbf{P}$ be a sustainable policy, and let $J_t(y_t) = P_{c,t}(y_t)$, for all t and all $y_t \in \mathcal{S}_t$. Then*

$$J_t(y_t) \leq \hat{T}_t(J_{t+1})(y_t), \tag{11}$$

for all t and all $y_t \in S_t$.

Moreover, if ${}_1\mathbf{J}$ that satisfies (11) and if for all t and all $y_t \in S_t$, the supremum in (10) is in fact a maximum, then ${}_1\mathbf{J}$ is in fact the consumption part of a sustainable policy.

Proof: To prove the necessity of the inequality, note that $\hat{T}(J_{t+1})$ is defined as the supremum over a set which must contain $P_{c,t}(y_t)$, since ${}_1\mathbf{P}$ is sustainable. The necessity part of the lemma is an immediate consequence of this observation.

To prove the sufficiency part of the lemma we construct the policy ${}_1\mathbf{P}$ that for all t and all $y_t \in S_t$ picks a maximal element of the set on the right hand side in (10). This policy is clearly sustainable. □

Assumption 2

$$\mu(J') \geq \mu(J'') \text{ for } J' \geq J'' \tag{12}$$

$$\hat{T}_t(\lim J_k) = \lim \hat{T}_t(J_k) \tag{13}$$

Theorem 5

$$J_t^*(y_t) = \hat{T}_t(J_{t+1}^*)(y_t) \tag{14}$$

for all t and all $y_t \in S_t$.

Proof: We first proove that $J_t^*(y_t) \leq \hat{T}_t(J_{t+1}^*)$, which is easily seen by comparing the set over wich the supremum is taken in (9) and (10). Let c_t be any consumption level such that there exists an $\tilde{y}$, with $(c_t, \tilde{y}) \in \sigma^*(y_t)$. Since $J_{t+1}^* \geq P_{c,t+1}$ for any sustainable policy, c_t is included in the set on the right hand side in (10). This proves the inequality.

To prove the opposite inequality, let $\epsilon > 0$ be given, and let P_{t+1}^k be a sequence of sustainable policies such that $\lim J_{t+1}^k = J_{t+1}^*$ with $J_{t+1}^k = P_{t+1}^k$. By construction of $\hat{T}$, there exists $(c_t^k, \tilde{y}^k) \in \sigma^*(y_t)$ with $c_t^k \geq \hat{T}(J_{t+1}^k) - \epsilon$. Without loss of generality, we may assume that the sequence c_t^k is convergent, then:

$$J_t^*(y_t) \geq \mathit{lim}\, c_t^k \geq \mathit{lim}\, \hat{T}_t(J_{t+1}^k)(y_t) - \epsilon = \hat{T}_t(J_{t+1}^*)(y_t) - \epsilon \tag{15}$$

Since this is true for any positive ϵ we have $J^*(y_t) \geq \hat{T}_t(J^*_{t+1})(y_t)$ □

In the following we will need this assumption:

Assumption 3 *If* $(c_t, \tilde{y}_{t+1}) \in \mathcal{T}_t(y_t)$ *so is also* $(c'_t, \tilde{y}_{t+1})$ *for all* $c'_t \leq c_t$.

This assumption looks like free disposal, but note that waste dispositions will show up in $\tilde{y}_{t+1}$. The assumption states that the transformation set defines an upper limit on the quality of life available given the bequest, but that it is possible to waist the resources and attain lower quality of life. Note also that policies that involves waste of resources will be suboptimal in an optimization problem, thus the effect of the assumption is mainly to simplify the set of sustainable quality of life.

Theorem 6 *If* ${}_1\mathcal{T}$ *satisfies assumption 3, then for any* $y_t \in \mathcal{S}_t$*, all* $c_t < J^*(y_t)$*,* $c_t \geq \underline{c}$ *are sustainable. Moreover, if for* $y_t \in \mathcal{S}_t$ *the set*

$$\psi_t(y_t) = \{(c_t, \mu(J^*(\tilde{y}_{t+1}))) \in \Re^2 : (c_t, \tilde{y}_{t+1}) \in \mathcal{T}_t(y_t)\}$$

is compact, then $c_t = J^*(y_t)$ *is also sustainable.*

Proof: The first part is a straightforward application of the previous assumption. The second part follows from the fact that the supremum on a compact set is a maximum. □

The characterization of the maximal sustainable SB, is a simple application of this theorem:

Corollary 1 *If for* $y_t \in \mathcal{S}_t$*,* $\psi_t(y_t)$ *is compact, the value of the maximal sustainable SB at* y_t *is given as*

$$\sigma(y_t) = \left\{(c_t, \tilde{y}_{t+1}) \in \mathcal{T}_t(y_t) : \underline{c} \leq c_t \leq J^*(y_t), \text{ and } \mu(J^*_{t+1}(\tilde{y}_{t+1})) \geq c_t\right\}. \quad (16)$$

An important requirement on a sustainable policy, that we have paid little attention to yet, is that there should be zero probability of a quality of life below the lower level $\underline{c}$. In the context of no risk, this requirement is not important, since if

$c_t \geq \underline{c}$, so is all c_s for $s \geq t$, along a sustainable path. With risk, we only require that the certainty equivalent of $\tilde{c}_{t+1}$ should not be less than c_t, but this allows for decreasing quality of life with positive probability. With an infinite horizon the obligation to avoid quality of life below this lower level, may be quite demanding. To study this we start by a trivial observation:

Lemma 6 *The maximum sustainable quality of life is nonincreasing as a function of* $\underline{c}$.

Proof This is obvious, since when $\underline{c}$ is increasing the set of feasible policies are decreasing. □

As pointed out above, we cannot exclude the possibility that the maximal sustainable SB is empty-valued at some states y_t. Recall that $\mathcal{S}_t$ is the set of inheritances at time t, that allows a sustainable resource management pair. If the the technology is stationary, then $\mathcal{S}_t = \mathcal{S}$, and is characterized by the following theorem.

Theorem 7 *$y \in \mathcal{S}$ if and only if there exists a resource management pair $(c, \tilde{y}) \in \mathcal{T}(y)$ with $c \geq \underline{c}$ so that supp$(\tilde{y}) \subseteq \mathcal{S}$.*

Proof: The only if part follows from the observation that $(c, \tilde{y}) \in \sigma^*(y) = T(\sigma^*)(y)$ implies $c \geq \underline{c}$, and $\sigma^*(y') \neq \emptyset$ for all $y' \in \text{supp}(\tilde{y})$. To prove the if part, it suffices to show that given the conditions of the theorem, there exists one sustainable resource management pair for any $y \in \mathcal{S}$. Combining the conditions in the theorem with assumption 3, we note that for all $y \in \mathcal{S}$ there is a resource management pair $(\underline{c}, \tilde{y}) \in \mathcal{T}(y)$ with supp$(\tilde{y}) \subseteq \mathcal{S}$. Construct σ by $\sigma(y) = (c, \tilde{y})$ if $y \in \mathcal{S}$, $\sigma(y) = \emptyset$ otherwise. Then ${}_1\sigma$ defined by $\sigma_t = \sigma$ for all t is clearly a nondecreasing SB □

Another charaterization of the set $\mathcal{S}_t$ can be derived by using minimax preferences. These are the preferences represented by the certainty equivalent $\hat{\mu}(\tilde{c}_{t+1}) = \min(\text{supp}(\tilde{c}_{t+1}))$. The corresponding maximal nondecreasing SB is denoted ${}_1\hat{\sigma}$, the maximum sustainable quality of life $\hat{J}(y_t)$, the set of inheritances allowing a sustainable policy $\hat{\mathcal{S}}_t$. Moreover $\hat{J}_0(y_t)$ denotes the maximum sustainable quality of life corresponding to $\hat{\mu}$ and $\underline{c} = 0$. Suppose that the real preferences are less pesimistic, $\mu(\tilde{c}_{t+1}) \geq \hat{\mu}(\tilde{c}_{t+1})$. Then we have the following theorem:

Theorem 8 *1. For all t and all $y_t \in Y_t$,*

$$\hat{\sigma}_t(y_t) \subseteq \sigma_t^*(y_t) \tag{17}$$

2. The set of inheritances that allows a sustainable resource management pair, is given as:

$$\mathcal{S}_t = \hat{\mathcal{S}}_t \subseteq \{y_t \in Y_t : \hat{J}_0(y_t) \geq \underline{c}\} \tag{18}$$

3. $\hat{J}(y_t)$ is independent of $\underline{c}$ as long as $\hat{J}(y_t) > \underline{c}$.

Proof: The first part of the theorem is obvious.

For the second claim, $\mathcal{S}_t \supseteq \hat{\mathcal{S}}_t$ is obvious from the first part of the theorem. To see that $\mathcal{S}_t \subseteq \hat{\mathcal{S}}_t$, remember that $J^*(y_t) \geq \underline{c}$, requires that there is a policy ${}_1P$ sustainable under μ with $P_{c,s}(y_s) \geq \underline{c}$ for all $s \geq t$ and $y_s \in S_{s,t}(y_t, {}_1P)$. It therefore follows from Assumption 3 that there exists a policy ${}_1\hat{P}$ sustainable under $\hat{\mu}$ with $\hat{P}_{c,s}(y_s) = \underline{c}$ for all $s \geq t$ and $y_s \in S_{s,t}(y_t, {}_1\hat{P})$. The last inclusion follows from Lemma 6.

To prove the last part of the theorem, note that if $\hat{J}(y_t) > \underline{c}$, there exists a policy ${}_1\hat{P}$, sustainable under $\hat{\mu}$ with $\hat{P}_{c,s}(y_s) > \underline{c}$ for all $s \geq t$ and $y_s \in S_{s,t}(y_t, {}_1\hat{P})$. □

The next theorem will be useful in the following examples.

Theorem 9 *Suppose $\underline{c} = 0$, and the technology and certainty equivalent is invariant to linear transformations, i.e., $(c_t, \tilde{y}_{t+1}) \in \mathcal{T}_t(y_t)$ implies that $(a \cdot c_t, a \cdot \tilde{y}_{t+1}) \in \mathcal{T}_t(a \cdot y_t)$ for any scalar $a > 0$, and $\mu(a \cdot \tilde{c}) = a \cdot \mu(\tilde{c})$, then is ${}_1\sigma^*$ is also invariant to linear transformations.*

Proof: Let $(c_t, \tilde{y}_{t+1}) \in \sigma_t^*(y_t)$. We then have to prove that $a \cdot (c_t, \tilde{y}_{t+1}) \in \sigma_t^*(a \cdot y_t)$. Let ${}_1\mathbf{P}$ be a sustainable policy, such that $P_t(y_t) = (c_t, \tilde{y}_{t+1})$. Define the policy ${}_1\hat{\mathbf{P}}$ as $\hat{P}_t(y_t) = aP_t(y_t/a)$. We claim that ${}_1\hat{\mathbf{P}}$ is sustainable. To see this note first that since ${}_1\mathbf{P}$ is sustainable, $P_{c,t}(y_t) \le \mu(P_{c,t+1}(P_{y,t}(y_t)))$ for all $y_t \in \mathcal{S}_t$ Then for $\frac{y_t}{a} \in \mathcal{S}_t$, we have:

$$\hat{P}_{c,t}(y_t) = aP_{c,t}(y_t/a) \le a\mu(P_{c,t+1}(P_{y,t}(y_t/a))) = \mu(\hat{P}_{c,t+1}(\hat{P}_{y,t}(y_t))). \qquad (19)$$

Since $\hat{P}_t(a \cdot y_t) = a \cdot (c_t, \tilde{y}_{t+1})$ we conclude $ay_t \in \mathcal{S}_t$, and $a \cdot (c_t, \tilde{y}_{t+1}) \in \sigma_t^*(a \cdot y_t)$ □

4 Two examples

4.1 Risky or safe technology

Consider the stationary and linear technology given by $(c_t, \tilde{y}_{t+1}) \in \mathcal{T}(y_t)$ if and only if

$$0 \le y_{t+1} = 2(y_t - c_t) \text{ with probability } 1 \qquad (20)$$

or

$$0 \le y_{t+1} = \begin{cases} (y_t - c_t) & \text{with probability } \frac{1}{2} \\ 4(y_t - c_t) & \text{with probability } \frac{1}{2} \end{cases} \qquad (21)$$

Thus, each generation must choose between a safe and a risky technique.

The level of subsistence is assumed to be equal to 0. Restrict attention to the class of von Neumann - Morgenstern (vNM) utility functions with constant relative risk aversion, meaning that the Arrow-Pratt measure of relative risk aversion, $u''(c)c/u'(c)$, is independent of c. The utility function with constant relative risk aversion α is

$$u(c) = \begin{cases} \frac{1}{1-\alpha}c^{1-\alpha} & \text{for } \alpha \neq 1; \alpha \ge 0 \\ ln(c) & \text{for } \alpha = 1 \end{cases}$$

In this case the maximal non-decreasing SB will be of the form $\sigma_t^*(y_t) = [0, J_t^*(y_t)]$. Since the technology is stationary and linear, the subsistence level is equal to 0, and the vNM utility function has constant relative risk aversion, the maximum sustainable quality of life must, according to theorem 9, be of the form

$$J_t^*(y_t) = x \cdot y_t,$$

and the problem of finding J^* may be solved by maximizing x constrained by feasibility and the SB being non-decreasing. Let $\bar{c}_t$ denote the maximal allowable consumption level at time t in the position y_t. Note that $\bar{c}_t = J(y_t) = x \cdot y_t$.

If only the safe technique were available, then,

$$\bar{c}_{t+1} = x \cdot y_{t+1} = x \cdot 2(y_t - c) = x \cdot 2(1-x)y_t.$$

Imposing that the SB be non-decreasing means that $\bar{c}_t = \bar{c}_{t+1}$ implying that $x = x \cdot 2(1-x)$ or $x = \frac{1}{2}$. Let J_S^* denote the maximal sustainable quality of life if only the safe technique were available, then $J_{St}^*(y_t) = \frac{1}{2}y_t$ independent of the degree of risk aversion.

If only the risky technique were available then

$$\bar{c}_{t+1} = x \cdot y_{t+1} = \begin{cases} x \cdot (y_t - \bar{c}_t) = x \cdot (1-x)y_t & \text{with probability } \frac{1}{2} \\ x \cdot 4(y_t - \bar{c}_t) = x \cdot 4(1-x)y_t & \text{with probability } \frac{1}{2}. \end{cases}$$

Imposing that the SB be non-decreasing means that $\bar{c}_t = \bar{c}_{t+1}$ implying, since the vNM utility functions has constant relative risk aversion, that

$$u(x) = \frac{1}{2}u(x \cdot (1-x)) + \frac{1}{2}u(x \cdot 4(1-x)).$$

Let J_R^* denote the maximal sustainable quality of life in this case. Then

$$\frac{1}{y_t}J_{Rt}^*(y_t) = \begin{cases} 1 - \left(\frac{2}{1+4^{1-\alpha}}\right)^{1/(1-\alpha)} & \text{for } \alpha \geq 0; \alpha \neq 1 \\ \frac{1}{2} & \text{for } \alpha = 1 \end{cases} \tag{22}$$

Arrow-Pratt measure of relative risk aversion equals 1, then $u(c) = \ln c$ and

$$\ln x = \frac{1}{2}\ln(x(1-x)) + \frac{1}{2}\ln(x \cdot 4(1-x)) = \ln(x \cdot 2(1-x))$$

or $x = \frac{1}{2}$. Hence, if only the risky technique were available and the Arrow-Pratt measure of relative risk aversion equals 1, then $\sigma_t^*(y_t) = [0, \frac{1}{2}sy_t]$.

It therefore follows that if both the safe and the risky technique are available, then the safe technique is used if the Arrow-Pratt measure of relative risk aversion exceeds 1, and the risky technique is used if the relative risk aversion is lower than 1. Thus

$$\frac{1}{y_t} J_t^*(y_t) = \frac{1}{y_t} \min(J_{Rt}^*(y_t), J_{St}^*(y_t)) = \begin{cases} 1 - \left(\frac{2}{1+4^{1-\alpha}}\right)^{1/(1-\alpha)} & \text{for } 1 > \alpha \geq 0 \\ \frac{1}{2} & \text{for } \alpha \geq 1 \end{cases} \tag{23}$$

4.2 Compensating possible irreversible changes

The next example demonstrates the use of theorem 5 and shows the impact of the lower limit $\underline{c}$ on quality of life.

Consider a society producing a single commodity and which is dependent upon environmental services. The commodity can either be used for consumption or investment, and is produced from capital with a linear technology. Suppose that there is a positive probability of an important degradation of environmental quality. For simplicity we will consider the case where this probability is independent of investments, thus the flow of environmental services will be exogenously given.

Let $x_t \geq 0$ be environmental services at time t, and suppose that

$$0 \leq x_{t+1} \leq \begin{cases} x_t & \text{with probability } 1-p \\ x_t \cdot a & \text{with probability } p \end{cases} \tag{24}$$

That is the environment is degraded to give a fraction $a < 1$ of the previous services, with probability p. The dynamic of capital expansion is given as

$$K_{t+1} = K_t(1+r) - C_t. \tag{25}$$

K_{t+1} is the capital at the end of period $t+1$, while $K_t(1+r)$ is the capital at the beginning of period t.

The quality of life is a geometric aggregate of the consumption of the commodity and of environmental services:

$$c_t = u(C_t, x_t) = C_t^{\alpha} x_t^{1-\alpha}, \tag{26}$$

where u is assumed to be a vonNeumann-Morgenstern utility, and hence the certainty equivalent to any lottery over c_t is equal to the expected value of c_t.

Theorem 10 *Suppose* $\underline{c} = 0$, *and* $1+r > (1-p(1-a^{1-\alpha}))^{-1/\alpha}$. *Then the maximum sustainable quality of life is*

$$J(K_t, x_t) = u(bK_t, x_t), \tag{27}$$

where b is given as

$$b = 1 + r - (1 - p(1 - a^{1-\alpha}))^{-1/\alpha} \tag{28}$$

Remark: When the assumption in the theorem is not satisfied, the right hand side of (28) would be non-positive. This corresponds to the case where sustainability is impossible, since the probability of degradation, or the size of degradations is to high to be compensated by savings.

Proof: Using theorem 5, we first have to prove that the proposed solution satisfies the optimality equation. We will prove a slightly stronger claim, that

$$J(K_t, x_t) = c_t = \mu(J(K_{t+1}, x_{t+1})), \tag{29}$$

for $J(K_t, x_t) = u(bK_t, x_t)$. The first part of this claim is obviously satisfied by choosing $C_t = bK_t$. For the second part, this can be written

$$b^{\alpha} K_t^{\alpha} x_t^{1-\alpha} = ((b(1 + r - b))^{\alpha} K_t^{\alpha})(p \cdot a^{1-\alpha} + (1 - p)) x_t^{1-\alpha} \tag{30}$$

This equation simplifies to

$$(1 + r - b)^{\alpha} \cdot (pa^{1-\alpha} + 1 - p) = 1, \tag{31}$$

which gives claimed value of b.

To prove that this is the maximum sustainable quality of life, we first note that by theorem 9, $J^*(\lambda K, \lambda x) = \lambda \cdot J^*(K, x)$, and obviously $J^*(K, \lambda x) = \lambda^{1-\alpha} J^*(K, x)$.

Hence, $J^*(K,x) = kK^\alpha x^{1-\alpha}$. If $J^*(K,x) > J(K,x)$, $k > b^\alpha$, but the previous calculation shows that no such k exists. □

We next charaterize the set of inheritances where a sustainable policy exists with $\underline{c} > 0$. Note that the minimax preferences can be derived by setting $p = 1$, thus by the previous theorem,

$$\hat{J}(K_t, x_t) = u(\hat{b}K_t, x_t), \tag{32}$$

where $\hat{b}$ is given as

$$\hat{b} = 1 + r - a^{(\alpha-1)/\alpha} \tag{33}$$

Using theorem 8, we easily derive the set of inheritances where a sustainable policy exists:

Corollary 2 *Given K_t, x_t, a sustainable policy exist for $\underline{c} > 0$, if and only if*

$$1 + r > a^{(\alpha-1)/\alpha} \tag{34}$$

and

$$\hat{J}(K_t, x_t) \geq \underline{c} \tag{35}$$

Note that the mere existence of a strictly positive positive subsistence level, requires that the requirement that b in (28) be positive, is replaced by the much stricter (34). We also note that J^* is not continous as a function of $\underline{c}$, at the point $\underline{c} = 0$. If b in (28) is positive, but the requirement in (34) is not met, then a sligh increase in $\underline{c}$ will leave all policies unsustainable, even at inheritances where $J^*(K_t, x_t)$ is very high for $\underline{c} = 0$.

It is interesting to compare the requirement to sustainability with the consumption that maximizes discounted utility. The next theorem gives the consumption strategy that maximizes discounted utility, with discount factor δ. The two constants $A = 1 - p(1 - a^{1-\alpha})$ and $b_m = (A\delta(1+r))^{1/(1-\alpha)} - 1$ are needed in the theorem.

Theorem 11 *Assume that* $(1+b_m)^\alpha < (A\delta)^{-1}$, *and* $r > b_m$. *Consider the problem of finding the consumption maximizing*

$$\bar{J}(K_t, x_t) = \max(E[\sum_{s=t}^{\infty} u(C_t, x_t)\delta^{s-t}] \tag{36}$$

subject to (25) and the No-Ponzy-Game condition

$$\lim_{t\to\infty} K_t(1+r)^{-t} \geq 0. \tag{37}$$

Then the optimal consumption is $C_t = b_m K_t$.

Proof: It is straightforward to verify that $\bar{J}$ given as

$$\bar{J}(K_t, x_t) = \left[\frac{r-b_m}{1+b_m}\right]^{\alpha-1} \frac{1}{A\delta}(K_t^\alpha x_t^{1-\alpha}) \tag{38}$$

solves the optimality equation. □

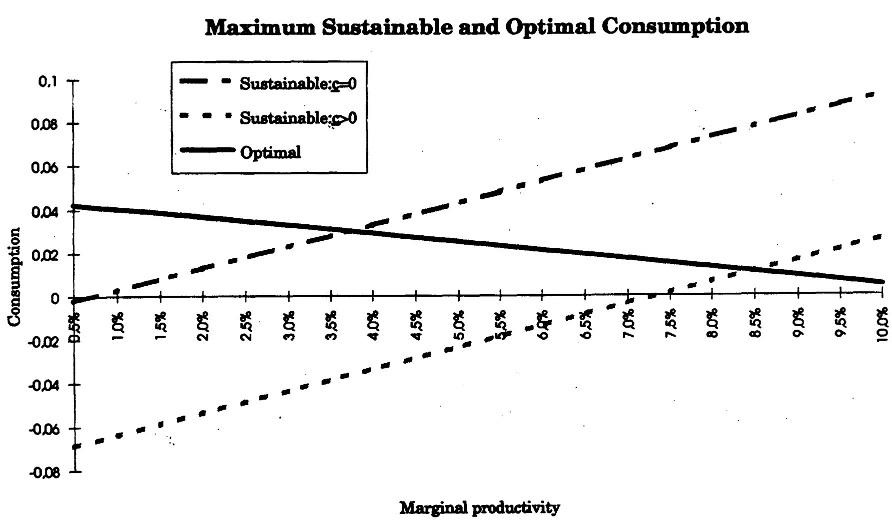

Figure 1: Sustainability requirement and optimal consumption as a function of marginal productivity

In Figure 1, the upper limit on consumption under the requirement of sustainability is given as a function of marginal productivity of capital. We consider two cases, $\underline{c} = 0$, and $\underline{c} \downarrow 0$. This is compared to the optimal consumption. Note that with increasing marginal productivity of capital, the requirement on sustainability is less restrictive, since it will be easier to compensate for potential environmental degradation. On the other hand, with increasing marginal productivity of capital, the return from savings will increase, making more saving a better option, hence the optimal consumption is a decreasing function of marginal productivity.

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