Discussion Paper

Central Bureau of Statistics, P.B. 8131 Dep, 0033 Oslo 1, Norway

No. 40 December 16, 1988

Optimal Starting and Stopping Rules for Resource Depletion when Price is Exogenous and Stochastic

by

Olav Bjerkholt and Kjell Arne Brekke

ABSTRACT

Optimal developement of exhaustible natural resources is analyzed under the assumption that oil prices follows a geometrical browninan motion. Under specific assumption on cost structure closed form soltions for the ***reservation price*** (the price above which start of depletion is optimal), the ***halt price*** (the price below which shut down is optimal), and the ***start/halt price*** (where no production is optimal below, and full capacity utilization is optimal above) are derived. From the theorems we derive rules of thumb for the impact of uncertainty and the impact of different types of costs. In the final section we demonstrate in a numerical model that our rules of thumb applies even to more general cases than those studied in the analytical part of the paper.

Paper presented at the Annual meeting of the Society for Economic Dynamics and Control, Tempe, Arizona, March 9-11, 1988.

1 Introduction

The value of an exhaustible natural resource reservoir such as a mine or an oil field depends upon the amount of resource, the cost of development and extraction, the future prices at which it can be sold and the rate of discount. Aspects of this problem were dealt with by two classical contributions to the natural resource literature: Gray (1914) discussed how to get the most out of a mine at given output price while Hotelling (1931) introduced the Hotelling price path, i.e. the price path at which the net price (resource rent) grows at exactly the rate of interest when unit costs and the rate of discount are constant. The Hotelling price path implies an astonishingly simple principle of evaluating exhaustible natural resource reservoirs, namely by multiplying the amount of resource with the current net price. This Hotelling valuation principle has been exploited by Miller and Upton (1985) for testing whether market values of oil and gas producing companies are consistent with the Hotelling price rule. It is argued that this valuation principle will have at least approximative validity under more general assumptions. Gray and Hotelling did not discuss the impact of uncertain future prices on optimal depletion and the market equilibrium. Miller and Upton (1985) indicates (pp. 10-11) that uncertainty *may* have considerable impact on the valuation. In this paper we shall replace the assumption of a known future price (which, of course, never holds) with another but more general simplification, namely that the price follows a path with known stochastic properties and look at the theoretical impact on valuation under optimal depletion.

The analysis is theoretical and the main purpose is to show that powerful methods of stochastic analysis can be exploited to give decision rules of thumb for a number of stylized problems related to resource extraction. The setting is that a known resource is in the ground and can be extracted at a known cost per unit. There may also be other cost components, such as an annual cost independent of the amount of production (*rental cost*). The price is supposed to follow a stochastic path with a trend component. Our knowledge about future prices is thus in the form of properties of its distribution. The price path is assumed continuous: the price today is the starting point for the price path in the near future. The mathematical representation of this idea is in the form of a stochastic differential equation.

The fundamental question in these problems is *when* to act: when to start extraction and when to stop. Hence, the analysis comes under the general heading of *optimal stopping problems*. The concept of *stopping time* is central in this analysis and is defined by way of introduction in section 2.

The solution for when to extract a natural resource once-and-for-all that follows from the Gray-Hotelling analysis, does not refer to the level of the price, but to its rate of increase. The increase in the net price is the return to holding the resource in the ground and with a deterministic increasing price the resource should be extracted according to the Hotelling rule when the momentary return to holding it in the ground is equal to the rate of interest (section 3).

But what when the price increase is known only in probability? The answer - wellknown in the literature - is that when the price is stochastic the resource owner should wait for a higher *reservation price* than in the deterministic case. The implication is also that the higher the variation around the trend, the higher should the reservation price be. An exact formula is given in section 4. However, this result is based on using the risk-free rate of interest for discounting the net return and the use of a properly risk adjusted rate of return may reverse the conclusion. The results of section 4 are also applicable with appropriate modifications when depletion takes time.

More realistic than a once-and-for-all extraction is an extraction process that takes time, a common assumption being that the amount extracted decreases exponentially over time. When extraction costs exceed the price there is a negative return to further extraction. With no chance of a reversal, extraction should be cut off immediately. But when the price is stochastic, there is always a chance that the price will pick up, so the question in this case is how low the price is allowed to fall before further extraction is permanently cut off. Again the answer differs from what it would be in the deterministic case. The solution is set out in section 5.1. Further clarification of how the answers depends upon cost structure is given in section 5.2.

Shutting-down a project before the resources are physically exhausted requires reconsideration of the reservation price of starting depletion. Section 6 considers the optimal start of an abandonable project and shows that the abandonment option *lowers* the reservation price.

A more intricate case is when the extraction process can be stopped and restarted. This will influence both when it is optimal to start extraction and when it is optimal to stop (temporarily). The solution to this case is given in section 7 and completes the theoretical results of this article.Included in the article is also a numerical example using real data for an off-shore field in the North Sea for which we demonstrate the kind of effects implied by our theoretical propositions.

The problem of evaluating natural resources under stochastic prices have received considerable attention in recent years. A proper evaluation is a precondition for answering optimal stopping problems as those posed in this article. Of particular importance for the approach taken in this paper are the highly seminal contributions of Pindyck (1980) and Brennan and Schwartz (1985).

2 Stopping times

In a deterministic context the problem of the optimal time to start or stop depletion of a natural resource, is solved by finding a deterministic point in time, say t^*, that fulfills certain optimality conditions. In stochastic models, however, the solution can not be restricted to be in the form of a deterministic point in time, since this would exclude the use of decision rules like: "Start depletion once the price reaches p^*". With such decision rules, the optimal time for starting depletion becomes stochastic if the price is stochastic. The problem is then to find the optimal time from a set of stochastic times. What is the relevant set of stochastic times?

We will introduce the concept of *stopping time*. Assume that the price P_t is a stochastic process. Then a stochastic time τ is a *stopping time* with respect to this process if we know whether $\tau \leq t$ when we know P_s for all $s \leq t$. If P_t is the only stochastic process in our model, and we assume that we can observe P_t at all times, then the set of stopping times with respect to P_t is the relevant set of stochastic times. This is so because any stopping time can be implemented, i.e. turned into a decision rule, while any stochastic time which is not a stopping time, would be impossible to implement, since it must depend on stochastic variables not specified in the model. Generally if X_t is a vector of all stochastic processes in a model, then the relevant set of stochastic times, is the set of stopping times with respect to X_t.

A typical example of a stopping time is the first exit time τ_G from a set G:

$$\tau_G = \inf\{t \geq 0 : X_t \notin G\}.$$

In this paper we consider tree kinds of optimal stopping problems. The first problem is to find the optimal time to start depletion of an oil field. In this case the optimal stopping time turns out to be of the form

$$\tau = \inf\{t \geq 0 : P_t \geq p^*\}$$

where p^* is called a *reservation price*. Another case is the optimal time for a shut-down of depletion, the optimal stopping time is then of the form

$$\tau = \inf\{t \geq 0 : P_t \leq p^*\}$$

where p^* now is called a *halt price*. Finally we consider the problem of optimal temporary halts in production. The solution turns out to be to produce whenever $P_t \geq p^*$, where p^* now is called a *start/halt price*.

3 Valuation under known future prices

We assume that our resource owner has a known amount of the resource which can be extracted at a constant unit cost $C > 0$. There are no restrictions on the rate of depletion. Furthermore, the future price path, P_t , is known and the rate of interest is r. The resource owner's problem is to choose which moment to empty his resource reservoir to maximize the present value of the resource. Nothing can be gained by gradual depletion. We might for this reason consider the unit resource rent rather than total rent. The problem can thus be stated

$$v(t) = \max_{\tau \geq t}(P_\tau - C)e^{-r\tau} \tag{1}$$

When the price path is monotonic and $t = \Theta(P)$ is the inverse of P_t ,the maximization problem can be reformulated as maximization with regard to a *reservation price*, p*:

$$V(P_t) = \max_{p^* \geq P_t}(p^* - C)e^{-r\Theta(p^*)} \tag{2}$$

Let us now consider an exponential growth path for prices:

$$\dot{P}_t = \alpha P_t \tag{3}$$

If $\alpha \geq r$ it is never optimal to extract, in the following we assume $0 < \alpha < r$. On this assumption (2) can be reformulated as

$$V(t,p) = \max_{p^* \geq p}(p^* - C)(\frac{p}{p^*})^{\frac{r}{\alpha}} e^{-rt} \tag{4}$$

which has the solution

$$p^* = \frac{r}{r-\alpha} C \tag{5}$$

unless $p > \frac{r}{r-\alpha}C$ in which case the solution is $p* = p$.

The underlying logic of the solution is that the resource should be extracted when the momentary return to holding the resource in the ground is equal to the rate of interest (unless this moment has already gone by). The unit value of the resource field is thus

$$V(t,p) = \frac{\alpha}{r-\alpha} C(\frac{p}{p^*})^{\frac{r}{\alpha}} e^{-rt} \tag{6}$$

4 Valuation when price is stochastic

Future prices are, of course, never known. A simple representation of a stochastic price is to assume that the price path is a diffusion process i.e. (3) is replaced by

$$dP_t = \alpha P_t dt + \sigma P_t dB_t \tag{7}$$

In (7) the first term express the exponential drift α of P_t while the second term represents the uncertainty of P_t as a Brownian motion (Wiener process) B_t. The path of P_t as given by (7) can loosely be described as an exponential path with some white noise added to it. In more technical terms P_t is said to follow a geometrical Brownian motion, which implies that:

$$E(P_s \mid P_t) = P_t e^{\alpha(s-t)} \text{ and } var(\ln(\frac{P_s}{P_t})) = \sigma^2(s-t) \tag{8}$$

The stochastic version of the problem in (1) is:

$$V(t,p) = \max_{\tau} E^{t,p}\{(P_\tau - C)e^{-r\tau}\} \tag{9}$$

where the maximum now is taken over the *set of stopping times with respect to* P_t. The solution is given in the following theorem:

Theorem 1 *Let*

$$V(t,p) = \max_{\tau} E^{t,p}\{(P_\tau - C)e^{-r\tau}\} \tag{10}$$

where:

$$dP_s = \alpha P_s ds + \sigma P_s dB_s. \tag{11}$$

Then

$$V(t,p) = \begin{cases} \frac{1}{\gamma-1}(p^*)^{-\gamma} C p^\gamma e^{-rt} & \textit{for } p < p^* \\ (p-C)e^{-rt} & \textit{otherwise} \end{cases} \tag{12}$$

where the reservation price is

$$p^* = \frac{\gamma}{\gamma - 1}C, \tag{13}$$

with

$$\gamma = \frac{1}{\sigma^2}[-(\alpha - \frac{1}{2}\sigma^2) + \sqrt{(\alpha - \frac{1}{2}\sigma^2)^2 + 2r\sigma^2}] > 1 \tag{14}$$

The optimal stopping rule is

$$\tau = inf\{t > 0 : P_t > p^*\} \tag{15}$$

or, simply, to wait until P_t is equal to the reservation price.

Theorem 1 is a wellknown result and a special case of the problem solved by McDonald and Siegel (1986).

(13) corresponds closely with the non-stochastic case, with γ entering instead of $\frac{r}{\alpha}$. From the expression for γ follows that

$$\lim_{\sigma^2 \to 0} \gamma = \frac{r}{\alpha} \tag{16}$$

which is the deterministic solution, cf. (5).

As σ increases over all bounds, γ will approach one from above and the reservation price will approach infinity. For given r the reservation price is higher under uncertainty than under certainty and increases with σ.

In the stochastic problem the appropriate discount rate is no longer the risk-free rate of return. Above we have implicitly assumed that r is the appropriate risk adjusted rate of return. This rate will generally depend upon σ. McDonald and Siegel (1986) shows this dependence by deriving an equivalent expression for γ:

$$\gamma = \frac{1}{\sigma^2}[-(\hat{\alpha} - \frac{1}{2}\sigma^2) + \sqrt{(\hat{\alpha} - \frac{1}{2}\sigma^2)^2 + 2r_F\sigma^2}] > 1 \tag{17}$$

where $\hat{\alpha} = \alpha - (r_P - r_F)$ and r_F is the risk-free rate of return. r_P is the required rate of return on an asset with the same uncertainty as P_t according to the Capital Asset Pricing Model. r_P can be shown to be a linear function of σ (assuming the correlation with the market portfolio constant). Hence the reservation price becomes a function of σ and is shown in figure 1 in the case of positive correlation with the market portfolio.

Proposition 1 *In the case of instantaneous extraction, and for a given r, the reservation price is higher under uncertainty than under certainty and increases with σ. Risk adjustment of r might reverse this effect for small σ.*

In (9) it is assumed that the resource is depleted instantly at the optimal point in time, i.e. the resource exploitation is started and stopped at the same time. Fortunately, the solution is valid for many other cases of resource exploitation. In general, this solution gives the optimal starting point for any project where the project value is of the form $(P_t - C)e^{-rt}$ with P_t following a geometrical Brownian motion. A project of gradual depletion of natural resources with a given profile depletion will be of this form.

To see this, let Q_s be the production at time s from start, and let C_s be variable unit costs, and K_s rental cost. There may also be an investment cost I at the start-up. The expected value of the project is then:

$$\begin{aligned} & E^{t,P_t}\{\int_0^T Q_s P_{t+s} e^{-r_P s} - (Q_s C_s - K_s)e^{-r_F s} ds - I\}e^{-rt} \\ &= [\int_0^T E^{t,P_t}\{P_{s+t} Q_s\} e^{-r_P s} ds - \int_0^T (Q_s C_s - K_s) e^{-r_F s} ds + I]e^{-rt} \\ &= [\int_0^T P_t e^{\alpha s} Q_s e^{-r_P s} ds - \overline{K}]e^{-rt} \\ &= (P_t \overline{Q} - \overline{K})e^{-rt} \\ &= \overline{Q}(P_t - \overline{C})e^{-rt} \end{aligned} \tag{18}$$

which is of the required form, if the price is geometrical Brownian. r_P is the rate of return corresponding to the risk on P_t, while r_F is the rate of return corresponding to the risk on costs, and r is the rate of return on the option. Note that since r_P is increasing with σ, $\overline{Q}$ will be decreasing with r_P, while $\overline{K}$ is unaffected, the generalized unit variable cost $\overline{C}$ will increase with σ. Hence, compared to the case of instantaneous extraction, an increase in σ will increase reservation price. This effect is stronger, the longer the lag between development decision and production is.

A special case of depletion profile is exponential depletion, i.e. production at a constant share, λ, of the remaining reserves, X_t, at any time. Using a common rate of return for both revenues and costs the expected value in this case is:

$$E^{t,p,x}\{\int_t^\infty [\lambda X_s(P_s - C) - K]e^{-rs} ds\} = (\frac{\lambda x p}{r + \lambda - \alpha} - \frac{\lambda x C}{r + \lambda} - \frac{K}{r})e^{-rt} \tag{19}$$

In the following sections exponential depletion will be assumed. We shall also study stochastic exponential depletion.

The results above may be summarized as:

Proposition 2 *The case of extraction over time with a given extraction path and with variable costs and other costs varying over time can be restated and cast in the form of instantaneous production. The reservation price does not depend on the type of cost, only upon the generalized unit variable cost $\overline{C}$, which in this case - in distinction from instantaneous extraction - will depend upon the required rate of return on the asset.*

5 Optimal stopping of exponential depletion

We consider now the exponential depletion of a given amount of resource. Unless all costs are capital costs accrued prior to the start of depletion the current net revenue may become negative. The question arises of when to stop depletion. Should it occur when net revenue is zero or at some level less than zero? The stop is assumed to be irreversible.

5.1 Optimal stopping with constant unit variable cost

Let X_t denote the resources remaining at time t. P_t is the price of extracted resource at time t, C unit cost, and λ the extraction rate.

The optimal stopping problem in this section is to find the *halt price* i.e. the price at which to stop extraction.

Theorem 2 *Assume*[1]

$$g^*(t,x,p) = \max_{\tau} E^{t,x,p}[\int_t^{\tau} \lambda X_s \cdot (P_s - C)e^{-rs}ds]. \tag{20}$$

exists, with

$$\begin{aligned} dX_t &= -\lambda X_t dt \\ dP_t &= \alpha P_t dt + \sigma P_t dB_t \end{aligned} \tag{21}$$

The solution is then :

$$\begin{aligned} g^*(t,x,p) &= x(\lambda[\tfrac{p}{r+\lambda-\alpha} - \tfrac{C}{r+\lambda}] + cp^{\mu})e^{-rt} \\ &= xf^*(p)e^{-rt} \end{aligned} \tag{22}$$

with

$$\mu = \frac{-(\alpha - \frac{1}{2}\sigma^2) - \sqrt{(\alpha - \frac{1}{2}\sigma^2)^2 + 2(r+\lambda)\sigma^2}}{\sigma^2} < 0 \tag{23}$$

and with

$$c = \frac{1}{1-\mu} \cdot \frac{\lambda C}{r+\lambda}(p^*)^{-\mu} \tag{24}$$

and the optimal stopping time is

$$\tau = \inf\{t > 0 : P_t < p^*\} \tag{25}$$

with:

$$p^* = (\frac{r+\lambda-\alpha}{r+\lambda}) \cdot (\frac{\mu}{\mu-1}) \cdot C \tag{26}$$

Proof:
From Theorem A 1, in the appendix, we know that g^* is a solution to the free boundary Dirichlet-problem:

$$\frac{\partial g^*}{\partial t} + \lambda x(p-C)e^{-rt} - \lambda x\frac{\partial g^*}{\partial x} + \alpha p \cdot \frac{\partial g^*}{\partial p} + \frac{1}{2}(\sigma p)^2 \frac{\partial^2 g^*}{\partial p^2} = 0 \tag{27}$$

for $(t,x,p) \in D$, and $\frac{\partial g^*}{\partial p} = 0$ for $(t,x,p) \in \delta D$. It is straightforward to check that g^* as defined in (22) satisfies this condition and, hence, is a candidate as solution. Note that:

$$\begin{aligned} g^*(t,x,p) &= \sup_{\tau} E^{t,x,p}[\int_t^{\tau} \lambda X_s(P_s - C)e^{-rs}ds] \\ &= \sup_{\tau} E^{t,x,p}[\int_t^{\tau} \lambda x e^{-\lambda(s-t)}(P_s - C)e^{-rs}ds] \\ &= e^{-rt}x \sup_{\tau} E^{t,p}[\int_0^{\tau} \lambda e^{-\lambda u}(P_u - C)e^{-ru}du] \\ &= e^{-rt}xf^*(p) \end{aligned} \tag{28}$$

where the equality between the second and the third equation is due to the strong Markov property of Itô diffusions.

Thus we knows that g^* must be decomposed in this specific way, and in addition it must solve the free boundary problem. Inserting the decomposition of g^* in the Dirichlet problem, we get an equation in f^*. It remains to prove the uniqueness of f^*. For this

[1] In this and later sections we will, for simplicity, use one common rate of return for both revenue and costs

we need a condition for $\lim_{p\to\infty} f^*(p)$. It seems reasonable to expect that the value of the stopping option will vanish as the price moves towards infinity. Hence if $xh(p)$ is the value of the resource without stopping option, the latter condition is:

$$\lim_{p\to\infty}[f^*(p) - h(p)] = 0 \tag{29}$$

where:

$$\begin{aligned} xh(p) &= E^{p,x}\{\textstyle\int_0^\infty \lambda X_t(P_t - C)e^{-rt}dt\} \\ &= \textstyle\int_0^\infty \lambda E^{p,x}X_t(P_t - C)e^{-rt}dt \\ &= \textstyle\int_0^\infty \lambda x e^{-\lambda t} p e^{\alpha t} e^{-rt}dt\} - \frac{x\lambda C}{r+\lambda}]. \\ &= x\lambda[\frac{p}{r+\lambda-\alpha} - \frac{C}{r+\lambda}]. \end{aligned} \tag{30}$$

Since f^* satisfy this condition, and since the solution to the differential equation with the corresponding bondary value restrictions is unique, f^* is the unique solution. **QED**

If $\sigma = 0$ we have a deterministic problem. In that case $\lambda X_s(P_s - C)$ is either positive – and then we shall never stop – or negative, but increasing. In the latter case it may be optimal to stop now, but it can never be optimal to stop later. In any case, if it is not optimal to halt immediately, the value of the field in the deterministic case is $\frac{\lambda x p}{r+\lambda-\alpha} - \frac{\lambda x C}{r+\lambda}$. Halt immediately is optimal if this expression is negative, and hence the halt price is $p^* = \frac{r+\lambda-\alpha}{r+\lambda}C$. From the solution given by (26) and (23) we conclude:

Proposition 3 *The halt price is less than unit cost C, and for a given discount rate the halt price is higher in the deterministic case than under uncertainty, i.e. $p^*_{\sigma=0} > p^*_{\sigma>0}$.*

5.2 Stochastic depletion, rental cost and abandonment cost

In the previous section we examined optimal stopping of depletion when there were only variable costs. In that case current costs approach zero as the production diminishes. Some costs, say rental cost for the field K, are independent of the level of production. There may also be a cost A of abandonment. A theorem which applies under these conditions follows. It is only slightly more general than a theorem in Olsen and Stensland (1987), which did not include abandonment cost. Our proof is, however, different. The result is included for completeness and because the expression for the value of the field is needed in section 6.

To analyze the impact of rental and abandonment cost, we disregard variable cost but assume that production is also stochastic.

Theorem 3 *Assume*

$$g^*(t,p,q) = \max_\tau E^{t,p,q}[\int_t^\tau (Q_s P_s - K)e^{-rs}ds - Ae^{-r\tau}] \tag{31}$$

exist with:

$$dQ_t = -\lambda Q_t dt + \beta Q_t dB_t^1 \tag{32}$$

$$dP_t = \alpha P_t dt + \sigma P_t dB_t^2 \tag{33}$$

and where $rA < K$, then

$$g^*(t,p,q) = [\frac{pq}{r+\lambda-\alpha} - \frac{K}{r} + c(pq)^\nu]e^{-rt} \tag{34}$$

with:

$$\nu = \frac{-(\alpha-\lambda-\frac{1}{2}(\sigma^2+\beta^2)) - \sqrt{(\alpha-\lambda-\frac{1}{2}(\sigma^2+\beta^2))^2 + 2r(\sigma^2+\beta^2)}}{(\sigma^2+\beta^2)} < 0 \tag{35}$$

and with:

$$c = (y^*)^{-\nu}\frac{1}{1-\nu}[\frac{K}{r} - A] \tag{36}$$

$$y^* = \frac{\nu}{\nu-1}\frac{r+\lambda-\alpha}{r}[K - rA] \tag{37}$$

The optimal plan is to stop production if $P_tQ_t \leq y^*$, i.e. $\tau = \inf\{t > 0 : P_tQ_t \leq y^*\}$, and $D = \{(t,p,q) : pq > y^*\}$.

Proof: g^* is a solution of the free boundary problem:

$$\frac{\partial g^*}{\partial t} + (pq-K)e^{-rt} + \alpha p\frac{\partial g^*}{\partial p} + \frac{1}{2}\sigma^2 p^2\frac{\partial^2 g^*}{\partial p^2} - \lambda q\frac{\partial g^*}{\partial q} + \frac{1}{2}\beta^2 q^2\frac{\partial^2 g^*}{\partial q^2} = 0 \tag{38}$$

for $(t,p,q) \in D$ where D is an open unknown set, called the continuation region, and where the boundary conditions are:

$$\begin{array}{ll} g^*(t,p,q) = -Ae^{-rt} & \text{for } (t,p,q) \in \delta D \\ \frac{\partial g^*}{\partial p}(t,p,q) = 0 & \text{for } (t,p,q) \in \delta D \\ \frac{\partial g^*}{\partial q}(t,p,q) = 0 & \text{for } (t,p,q) \in \delta D \end{array} \tag{39}$$

It is straightforward to check that g^* satisfies this equation, and it remains to show uniqueness. This is done as in the proof of theorem 1, since the value of the resource without abandonment option is:

$$[\frac{pq}{r+\lambda-\alpha} - \frac{K}{r}]e^{-rt} \tag{40}$$

According to Theorem A 1, in the appendix, g^* is the solution to the problem. **QED**

In this problem the stopping criterion is the income flow y^*, rather than a halt price. But for a given extraction Q_t the halt price is

$$p^* = y^*/Q_t$$

Since $\nu < 0$, y^* is less than $\frac{r+\lambda-\alpha}{r}[K - rA]$ which is the halt income under certainty, when $\alpha \geq \lambda$. If $\lambda > \alpha$ the halt income under certainty is $K - rA$, but in this case $\frac{\nu}{\nu-1} < \frac{r}{r+\lambda-\alpha}$, and hence $y^* < [K - rA]$. We conclude:

Proposition 4 *The halt price under uncertainty is less than under certainty. Furthermore, it is increasing with rental cost, decreasing with abandonment cost and decreasing in size of remaining reserves.*

We can prove that this conclusion holds also for the appropriate risk adjusted discount rate (if $r_P \geq r_F$).

6 Optimal start of an abandonable project

In section 4 we noted that if the price is geometrical Brownian, any given depletion program, e.g. exponential depletion, has a project value of the form $(V_t - C)e^{-rt}$ with V_t geometrical Brownian. However, if we have the option to abandon the project, this is no longer the case. To study this we can use the result from section 5.2 where we found the value of a field in the case of stochastic exponential depletion and an abandonment option. We use the same cost assumption as in that section. We assume that we get no new information about the extraction path as long as the extraction is not started. The depletion level Q_0 at production start is thus known with certainty. Let $\overline{p}$ denote the halt price at production start. Then the project value will be:

$$V(t, P_t) = \begin{cases} -Ae^{-rt} & \text{for } P_t < \overline{p} \\ (\varsigma P_t - \frac{K}{r} + \phi P_t^{\nu})e^{-rt} & \text{otherwise} \end{cases} \tag{41}$$

where $\varsigma = \frac{\lambda Q_0}{r+\lambda-\alpha}$, and $\phi = (p^*)^{-\nu}\frac{1}{1-\nu}[\frac{K}{r} - A]$

We will try to find the optimal starting rule in this case. Obviously it is not optimal to start the project if it will be immediately abandoned, hence the reservation price p^* must satisfy $p^* > \overline{p}$. We might, for sake of convenience, argue as if the project value for all P_t is given by:

$$g(t, P_t) = (\varsigma P_t - \frac{K}{r} + \phi P_t^{\nu})e^{-rt} \tag{42}$$

Theorem 4 *The solution to:*

$$g^*(t,p) = \max_{\tau} E^{t,p}[g(\tau, P_\tau)] \tag{43}$$

subject to

$$dP_t = \alpha P_t dt + \sigma P_t dB_t \tag{44}$$

is:

$$g^*(t,p) = cp^{\gamma}e^{-rt} \tag{45}$$

where

$$\gamma = \frac{-(\alpha - \frac{1}{2}\sigma^2) + \sqrt{(\alpha - \frac{1}{2}\sigma^2)^2 + 2r\sigma^2}}{\sigma^2} > 1 \tag{46}$$

$$c = (p^*)^{-\gamma}\frac{1}{\gamma - 1}[\frac{K}{r} - (1-\nu)\phi(p^*)^{\nu}] \tag{47}$$

and p^* is determined by

$$\varsigma(\gamma - 1)p^* = \frac{\gamma K}{r} - \frac{\gamma - \nu}{1-\nu}[\frac{K}{r} - A] \tag{48}$$

The optimal stopping rule is

$$\tau = \inf\{t > 0 : P_t > p^*\} \tag{49}$$

Proof:
The solution is a function $g^*(t,p) = h(p)e^{-rt}$ satisfying the free boundary problem:

$$-rh + \alpha ph' + \frac{1}{2}(\sigma p)^2 h'' = 0 \tag{50}$$

for $p < p^*$, and with $h(0) = 0$, $h(p^*) = f(p^*)$ and $h'(p^*) = f'(p^*)$. g^* is a solution to this system, and it is unique. **QED**

Using the notation from (18) in section 4, $\varsigma = \overline{Q}$, $\frac{K}{r} = \overline{K}$ and $\overline{C} = \frac{\overline{K}}{Q}$. Define $\overline{A} = \frac{A}{\varsigma}$ we get:

$$p^* = \frac{\gamma}{\gamma - 1}\overline{C} - \frac{\gamma - \nu}{(1-\nu)(\gamma - 1)}[\overline{C} - \overline{A}]. \tag{51}$$

Comparing with (13) and noting that by assumption $\overline{C} > \overline{A}$, we find that the abandonment option reduces the reservation price.

Proposition 5 *The reservation price with the abandonment option is less than the reservation price without such an option.*

7 Stopping and restarting

In the section 5 we discussed the optimal time of halting the depletion, but disregarded the possibility of restarting, i.e. reopening a field when prices increase sufficiently to make it worthwhile. We will now briefly consider the restarting problem assuming that extraction can be started and stopped an arbitrary number of times without additional cost. This problem is dealt with by Brennan and Schwartz (1985) who also consider the costs of opening and closing the mine.

Theorem 5 *Let:*

$$h^*(t,x,p) = \sup_{\eta} E^{t,x,p}\{\int_t^\infty [\eta X_s \cdot (P_s - C) - K]e^{-rs}ds\} \tag{52}$$

subject to:

$$\begin{aligned} dP_t &= \alpha \cdot P_t dt + \sigma \cdot P_t dB_t \\ dX_t &= -\eta X_t dt \\ \eta &\in [0, \lambda] \end{aligned} \tag{53}$$

Let further:

$$h(t,x,p) = [xF(p) - \frac{K}{r}]e^{-rt} \tag{54}$$

with

$$F(p) = \begin{cases} F_1(p) & \text{for } p < p^* \\ F_2(p) & \text{otherwise} \end{cases} \tag{55}$$

where

$$F_1(p) = c_1 p^\gamma \tag{56}$$

$$F_2(p) = \lambda[\frac{p}{r+\lambda-\alpha} - \frac{C}{r+\lambda}] + c_2 p^{\mu} \tag{57}$$

and

$$\begin{aligned} c_1 &= (p^*)^{-\gamma}\tfrac{\lambda}{\lambda+r}\cdot\tfrac{-\mu}{(\gamma-\mu)(\gamma-1)}\cdot C \\ c_2 &= (p^*)^{-\mu}\tfrac{\lambda}{\lambda+r}\cdot\tfrac{-\gamma}{(\gamma-\mu)(\mu-1)}\cdot C \end{aligned} \tag{58}$$

$$p^* = \frac{\gamma}{\gamma-1}\cdot\frac{\lambda+r-\alpha}{\lambda+r}\cdot\frac{-\mu}{1-\mu}\cdot C \tag{59}$$

and

$$\begin{aligned} \gamma &= \tfrac{1}{\sigma^2}[-(\alpha-\tfrac{1}{2}\sigma^2)+\sqrt{(\alpha-\tfrac{1}{2}\sigma^2)^2+2r\sigma^2}] \\ \mu &= \tfrac{1}{\sigma^2}[-(\alpha-\tfrac{1}{2}\sigma^2)-\sqrt{(\alpha-\tfrac{1}{2}\sigma^2)^2+2(r+\lambda)\sigma^2}] \end{aligned} \tag{60}$$

Then $h^* = h$ and the optimal control is

$$\eta^* = \begin{cases} \lambda & \textit{for } p > p^* \\ 0 & \textit{otherwise} \end{cases} \tag{61}$$

Proof:

The Hamilton-Jacobi-Bellman equation is:

$$\sup_{\eta}[[\eta x(p-C)-K]e^{-rt} + \frac{\partial h}{\partial t} - \eta x\frac{\partial h}{\partial x} + \alpha p\frac{\partial h}{\partial p} + \frac{1}{2}(\sigma p)^2\frac{\partial^2 h}{\partial p^2}] = 0 \tag{62}$$

with boundary conditions: $h(x,0,t) = -\frac{K}{r}e^{-rt} = h(0,p,t)$.

Assume that $h^* = h$, it is then easy to check that supremum is attained by η^*. Furthermore it is straightforward to check that h in fact satisfy the Hamilton-Jacobi-Bellman equation. Finally we have:

$$\begin{aligned} F_1(p^*) &= F_2(p^*) \\ F_1'(p^*) &= F_2'(p^*) \\ F_1''(p^*) &= F_2''(p^*) \end{aligned} \tag{63}$$

This implies that h is C^2 which is a sufficient condition for $h^* = h$ **QED**

Since p^* in (59) is independent of rental cost, K, we conclude:

Proposition 6 *The start/halt price with the restarting option is independent of rental cost.*

In section 5.1 we studied the optimal stopping of depletion in a model with no rental cost. By comparison of (26) with (59) we find that the start/halt price is equal to the halt price times the factor $\frac{\gamma}{\gamma-1} > 1$ (since $\gamma > 0$), hence

Proposition 7 *For zero rental cost the start/halt price is higher than the halt price.*

8 The impact of different types of cost on halting decision; an application to the Ula field of the North Sea.

Let us review our results so far.

We found in proposition 2 that the reservation price for development of a field is increasing in costs, but independent of the type of cost, only aggregate costs matters. In the abandonable case of proposition 5, the dependence on type of costs is more complex.

In proposition 3 we found that the halt price is increasing with variable unit costs, and according to proposition 4 it is increasing with rental costs as well, while it is decreasing with abandonment cost and the size of remaining reserves. All this is reasonable, since an abandonment will save future unit cost, and future rental cost. An early abandonment will, however, increase present value of abandonment costs.

In the case of stopping and restarting, we found in proposition 6 that the optimal decision is not affected by rental costs. This is reasonable, since a temporary halt does not save any rental costs. The start/halt price is, however, increasing in variable unit cost. Furthermore for zero rental cost we found in proposition 7 that the start/halt price is higher than the halt price in the abandonment case.

The analysis of the previous models gives an indication of what we might expect in a case where all cost components are nonzero, and with an option of abandonment as well as an option of temporary halts. Consider a field where development has already started. From the previous analysis we might expect that for small rental cost the start/halt price will dominate the halt price (Prop. 7). Higher rental cost would imply higher halt price (Prop 4) while the start/halt price will remain approximately unchanged (Prop. 6). A reduction in remaining reserves has the same effect (Prop. 4). Eventually the halt price dominates the start/halt price and temporary halt will never be optimal. (Our analysis this far have excluded stopping and restarting costs, but such costs obviously makes the temporary halt option even less attractive.)

To check these assertions we have constructed a discrete time/state model which is numerically solvable. Let:

$$V(t,k,p,l) = \sup_{a_t} E^{t,k,p,l}\{\sum_{t}^{T}[\pi(s,k,P_s,l_s,a_s)\delta^{-s}\} \tag{64}$$

be the value of an oil field at time t, after $k-1$ years of production. π is annual profit, p is the current price, and l indicate the state of the field, and a_t is the action chosen.

$$l = \begin{cases} 1 & \text{open} \\ 0 & \text{closed} \\ -1 & \text{abandoned} \end{cases} \tag{65}$$

$$a = \begin{cases} 1 & \text{produce} \\ 0 & \text{halt} \\ -1 & \text{abandon} \end{cases} \tag{66}$$

C	C_o	C_c	K	A	T	δ
6.95 $/barrel	5 mill $	0	60 mill $/year	137.5 mill $	30 year	1.05

Table 1: Data for Ula

In period T, a still operating field must be abandoned. π is defined as:

$$\pi(t,k,p,l,a) = \begin{cases} (p-C)Q(k) - K & \text{for } a=1, l=1 \\ -C_c - K & \text{for } a=0, l=1 \\ (p-C)Q(k) - K - C_o & \text{for } a=1, l=0 \\ -K & \text{for } a=0, l=0 \\ 0 & \text{for } l=-1 \\ -A & \text{for } a=-1, l\neq -1 \end{cases} \tag{67}$$

where C is unit variable cost, K is rental cost, A is the cost of abandonment, C_c the cost of closing the field, and C_o the cost of opening it. Obviously:

$$k_t = \begin{cases} k_{t-1}+1 & \text{for } a_t = 1 \\ k_{t-1} & \text{else} \end{cases}$$

and

$$l_t = \begin{cases} l_{t-1} & \text{for } l_{t-1} = -1 \\ a_t & \text{else} \end{cases}$$

By dynamic programming

$$V(t,k,p,l) = \max_a [\pi(t,k,p,l,a) + \delta^{-1} E^{t,k,p,l} V(t+1, k_{t+1}(a), P_{t+1}, l_{t+1}(a))] \tag{68}$$

with $V(T,\cdot,\cdot,\cdot)$ known.

For a finite number of possible prices, V can be represented in numerical form on a computer. It is then straightforward to compute V, and hence the optimal strategy, for $t = T, T-1, ..., 0$ successively, using dynamic programming.

We have applied this algorithm to data for the Norwegian oil field Ula in the North Sea. Ula is a small field with a peak production of 78.45 thousand barrel a day in 1988, with 95% (in o.e.) oil and the rest is natural gas. The relevant data is

The optimal solution for $l = 1$, with data from Ula is shown in figure 3 a). This is compared to a case with rental costs reduced from $ 60 mill pr. year, in a) to $ 30 mill pr. year in b). As expected from the previous analysis the start/halt price is relatively unchanged at about 8-10 $/barrel as rental cost is decreasing from $ 60 mill pr. year to $ 30 mill pr year, and as the size of remaining reserves decreases through production. The halt price, however, is sharply decreased by the decrease in rental cost, and it increases with year of production as the size of remaining reserves decreases.

This analysis shows that the even though the propositions was proved only under rather strong assumption, they render valid rules of thumb even under more complex cost structures.

A An optimal stop theorem

Let

$$dX_t = b(t, X_t)\,dt + \sigma(t, X_t)\,dB_t \tag{69}$$

be an n-dimensional Itô diffusion. We define the operator

$$(\mathcal{A}f)(t,x) = \frac{\partial f}{\partial t} + \sum_i b_i(t,x)\frac{\partial f}{\partial x_i} + \frac{1}{2}\sum_{i,j}(\sigma\sigma^T)_{i,j}(t,x)\frac{\partial^2 f}{\partial x_i \partial x_j} \tag{70}$$

for $f \in C^2(R^n)$.

The following theorem is a special case of Shiryayev (1978, Theorem 3.17).

Theorem A 1 *Let*

$$g^*(t,x) = \sup_\tau E^{t,x}\{\int_t^\tau f(s, X_s)ds + g(\tau, X_\tau)\} \tag{71}$$

with X_t a n-dimensional geometrical Brownian motion:

$$\frac{dX_{it}}{X_{it}} = a_i dt + \sum_{j=1}^m b_{ij} dB_{jt} \tag{72}$$

Let

$$D = \{(t,x) : g(t,x) < g^*(t,x)\} \tag{73}$$

Assume that $g \in C^1$, $g^*(t,x) \in C^2(D)$ and that $\frac{\partial g^*}{\partial x_i}$ exists on δD, at least as a one sided differential from D. Assume further that

$$\sup_{(t,x)\in\delta D} g(t,x) < M$$

then

$$(\mathcal{A}g^*)(t,x) = -f(t,x) \text{ for } (t,x) \in D \tag{74}$$

$$g^*(t,x) = g(t,x) \text{ for all } (t,x) \in \delta D \tag{75}$$

and for all i

$$\frac{\partial g^*}{\partial x_i}(t,x) = \frac{\partial g}{\partial x_i}(t,x) \text{ for all } (t,x) \in \delta D \tag{76}$$

To use Shiryayev's theorem we set $g^M(t,x) = max(M, g(t,x))$, since obviously $g^{M*} = g^*$ in D. Furthermore Shiryayev's theorem is stated for the case $f = 0$. However, the extension to a general f is straightforward, introducing a new state variable $X_{n+1,t}$ where:

$$dX_{n+1,t} = f(t, X_t)dt \tag{77}$$

References

Brennan, M.J. and E.S. Schwartz (1985): Evaluating Natural Resource Investments. *Journal of Business, 58, no. 2*, 135-157.

Gray, L.C. (1914): Rent under the Assumption of Exhaustibility. *Quarterly Journal of Economics*, 28, 466-489.

Hotelling, H. (1931): The Economics of Exhaustible Resources. *Journal of Political Economy*, 39, 137-175.

McDonald, R. and D. Siegel (1986): The Value of Waiting to Invest. *Quarterly Journal of Economics*, Nov. 1986.

Miller, M.H. and C.W. Upton (1985): A Test of the Hotelling Valuation Principle. *Journal of Political Economy, 93, no. 1*, 1-25.

Øksendal, B. (1985): *Stochastic Differential Equations; An Introduction with applications* Springer-Verlag.

Olsen, T. and G. Stensland (1988): "Optimal Shut Down Decisions in Resource Extraction", *Economic Letters*, Forthcoming.

Pindyck, R.S. (1980): Uncertainty and Exhaustible Resource Markets. *Journal of Political Economy, 88, no. 6*, 1203-1225.

Shiryayev, A.N. (1978): *Optimal Stopping Rules* Springer-Verlag.

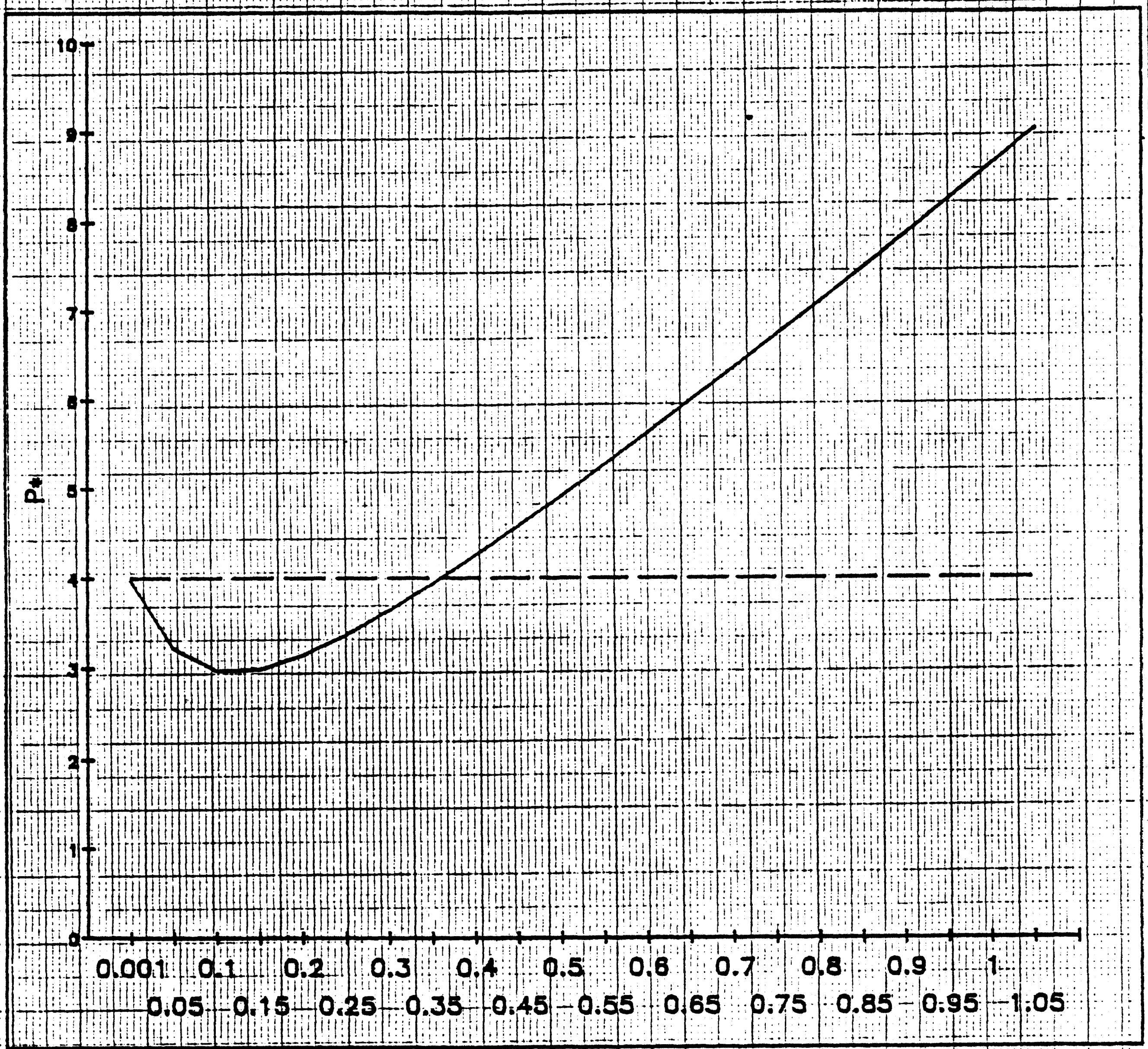

Figure 1: The reservation price as a function of σ.

a)

b)

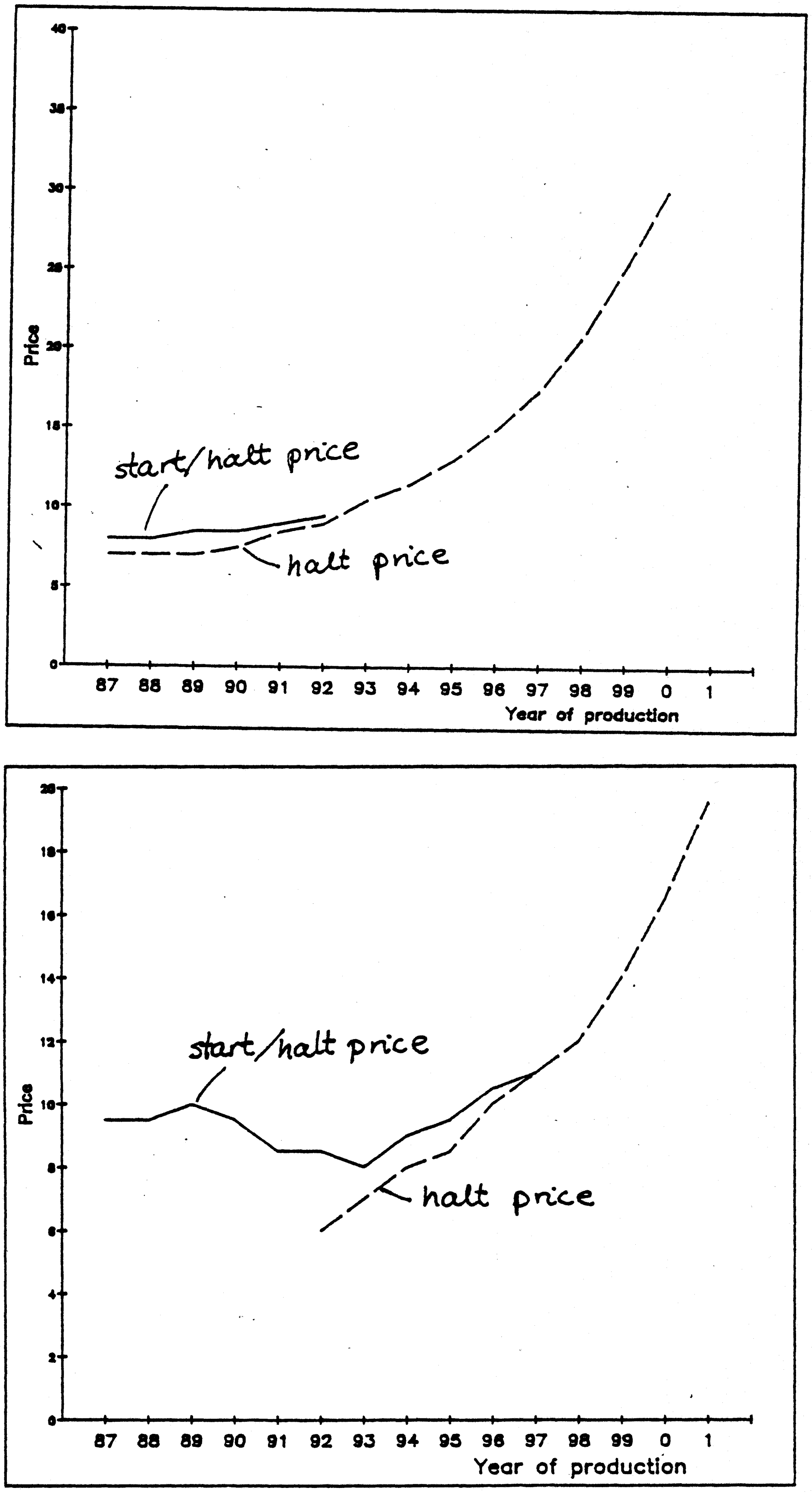

Figure 2: Optimal stop/abandonment on Ula. Production independent cost $ 60 mill pr year in a) and $ 30 mill pr year in b)

ISSUED IN THE SERIES DISCUSSION PAPER

No. 1 I. Aslaksen and O. Bjerkholt: Certainty Equivalence Procedures in the Macroeconomic Planning of an Oil Economy.

No. 3 E. Biørn: On the Prediction of Population Totals from Sample surveys Based on Rotating Panels.

No. 4 P. Frenger: A Short Run Dynamic Equilibrium Model of the Norwegian Prduction Sectors.

No. 5 I. Aslaksen and O. Bjerkholt: Certainty Equivalence Procedures in Decision-Making under Uncertainty: an Empirical Application.

No. 6 E. Biørn: Depreciation Profiles and the User Cost of Capital.

No. 7 P. Frenger: A Directional Shadow Elasticity of Substitution.

No. 8 S. Longva, L. Lorentsen, and Ø. Olsen: The Multi-Sectoral Model MSG-4, Formal Structure and Empirical Characteristics.

No. 9 J. Fagerberg and G. Sollie: The Method of Constant Market Shares Revisited.

No.10 E. Biørn: Specification of Consumer Demand Models with Stocahstic Elements in the Utility Function and the first Order Conditions.

No.11 E. Biørn, E. Holmøy, and Ø. Olsen: Gross and Net Capital, Productivity and the form of the Survival Function . Some Norwegian Evidence.

No.12 J. K. Dagsvik: Markov Chains Generated by Maximizing Components of Multidimensional Extremal Processes.

No.13 E. Biørn, M. Jensen, and M. Reymert: KVARTS - A Quarterly Model of the Norwegian Economy.

No.14 R. Aaberge: On the Problem of Measuring Inequality.

No.15 A-M. Jensen and T. Schweder: The Engine of Fertility - Influenced by Interbirth Employment.

No.16 E. Biørn: Energy Price Changes, and Induced Scrapping and Revaluation of Capital - A Putty-Clay Approach.

No.17 E. Biørn and P. Frenger: Expectations, Substitution, and Scrapping in a Putty-Clay Model.

No.18 R. Bergan, Å. Cappelen, S. Longva, and N. M. Stølen: MODAG A - A Medium Term Annual Macroeconomic Model of the Norwegian Economy.

No.19 E. Biørn and H. Olsen: A Generalized Single Equation Error Correction Model and its Application to Quarterly Data.

No.20 K. H. Alfsen, D. A. Hanson, and S. Glomsrød: Direct and Indirect Effects of reducing SO_2 Emissions: Experimental Calculations of the MSG-4E Model.

No.21 J. K. Dagsvik: Econometric Analysis of Labor Supply in a Life Cycle Context with Uncertainty.

No.22 K. A. Brekke, E. Gjelsvik, B. H. Vatne: A Dynamic Supply Side Game Applied to the European Gas Market.

No.23 S. Bartlett, J. K. Dagsvik, Ø. Olsen and S. Strøm: Fuel Choice and the Demand for Natural Gas in Western European Households.

No.24 J. K. Dagsvik and R. Aaberge: Stochastic Properties and Functional Forms in Life Cycle Models for Transitions into and out of Employment.

No.25 T. J. Klette: Taxing or Subsidising an Exporting Industry.

No.26 K. J. Berger, O. Bjerkholt and Ø. Olsen: What are the Options for non-OPEC Producing Countries.

No.27 A. Aaheim: Depletion of Large Gas Fields with Thin Oil Layers and Uncertain Stocks.

No.28 J. K. Dagsvik: A Modification of Heckman's Two Stage Estimation Procedure that is Applicable when the Budget Set is Convex.

No.29 K. Berger, Å. Cappelen and I. Svendsen: Investment Booms in an Oil Economy - The Norwegian Case.

No.30 A. Rygh Swensen: Estimating Change in a Proportion by Combining Measurements from a True and a Fallible Classifier.

No.31 J.K. Dagsvik: The Continuous Generalized Extreme Value Model with Special Reference to Static Models of Labor Supply.

No.32 K. Berger, M. Hoel, S. Holden and Ø. Olsen: The Oil Market as an Oligopoly.

No.33 I.A.K. Anderson, J.K. Dagsvik, S. Strøm and T. Wennemo: Non-Convex Budget Set, Hours Restrictions and Labor Supply in Sweden.

No.34 E. Holmøy and Ø. Olsen: A Note on Myopic Decision Rules in the Neoclassical Theory of Producer Behaviour, 1988.

No.35 E. Biørn and H. Olsen: Production - Demand Adjustment in Norwegian Manufacturing: A Quarterly Error Correction Model, 1988.

No.36 J. K. Dagsvik and S. Strøm: A Labor Supply Model for Married Couples with Non-Convex Budget Sets and Latent Rationing, 1988.

No.37 T. Skoglund and A. Stokka: Problems of Linking Single-Region and Multiregional Economic Models, 1988.

No.38 T. J. Klette: The Norwegian Aluminium industry, Electricity prices and Welfare,1988

No.39 I. Aslaksen, O. Bjerkholt and K. A. Brekke: Optimal Sequencing of Hydroelectric and Thermal Power Generation under Energy Price Uncertainty and Demand Fluctuations, 1988.

No.40 O. Bjerkholt and K.A. Brekke: Optimal Starting and Stopping Rules for Resource Depletion when Price is Exogenous and Stochastic, 1988.