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# Fitting Curves to Age-Specific Fertility Rates:

## Some Examples

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CONTENTS



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## 1. Introduction

1A. While one would expect fertility to be a fairly smooth function of age, the diagram of a set of observed age-specific fertility rates tend to show a rather rugged curve, particularly in small populations. (See e.g. figure 4 below.) When it is reasonable to ascribe such "irregularities" to accidential circumstances, one may get a better picture of the underlying fertility by fitting some nice mathematical function to the observed values.

There are a large number of functions which could be fitted to observed age-specific fertility rates. In this paper we will give examples of the fitting properties of some of these.

1B. The problem arose in connection with the making of population projections for Norwegian regions (Gilje, 1968). As the fertility can vary appreciably between the various parts of our widespread country, we wanted separate fertility measures for each of a number of small regions. The observed fertility turned out to fluctuate quite a lot with age, and more in regions with a small population than in regions with a larger.

We have no reason to believe that these "irregularities" are due to particular tendencies in the birth habits of the observed stock of women. Therefore we ascribe them to chance fluctuations, and a priori assume that the underlying fertility of the population in such regions is a regular function of some kind. This assumption should justify that we try to fit a mathematical function to the estimates, particularly for small populations. Here we of course must use some discretion. When a population is too small, the chance variations will predominate, and a curve fitted to such observed values have of course little or no interest.

## 2. Some previous papers related to this problem

2A. Investigations on curve fitting to age-specific fertility rates have recently been made by Mitra (1967), by Keyfitz (1967), and by Tekse (1967). The latter two give references to many earlier authors.

In this paper, we shall take the studies by Tekse (1967) and Yntema (1953, 1956, and elsewhere) as our starting point.

2B. Yntema recommends using the following function, originally suggested by Hadwiger (1940, 1941):

$$(2.1) \qquad h_y(x) = \frac{ab}{c\sqrt{\pi}} \left(\frac{c}{x}\right)^{\frac{3}{2}} \exp\{-b^2(\frac{c}{x} + \frac{x}{c} - 2)\}.$$

Here x represents age attained, while a, b and c are parameters to be fitted. If we let $\hat{f}_x$ denote the observed fertility rate at age x, $\hat{R}_0 = \sum_x \hat{f}_x$ the corresponding observed gross fertility rate, $\hat{R}_k = \sum_x x^k \hat{f}_x$, and $\hat{T} = \hat{R}_1/\hat{R}_0$, Yntema would estimate the parameters a and c by

(2.2) $\hat{a} = \hat{R}_0$ and $\hat{c} = \hat{T}$,

respectively. His suggestions for estimators for b amount to

(2.3) $\hat{b}_1 = \{\tfrac{1}{2}\hat{R}_1^2/(\hat{R}_0\,\hat{R}_2 - \hat{R}_1^2)\}^{\frac{1}{2}}$, $\hat{b}_2 = \hat{T}\sqrt{\pi}\hat{f}_{\check{T}}/\hat{R}_0$,

as well as

(2.4) $\hat{b} = \tfrac{1}{2}(\hat{b}_1+\hat{b}_2)$

with a preference for $\hat{b}$. Here $\check{T}$ is the integer value obtained by rounding of $\hat{T}$. The estimators $\hat{a}$, $\hat{b}_1$, and $\hat{c}$ have been found by the method of moments.

Yntema (1953) also investigates a γ-function of the form

(2.5) $\gamma_y(x) = a(1 + \frac{x+d}{b})^{bc} \exp\{-c(x+d)\}$

as well as that of a normal curve. He considers the corresponding fit to this Dutch data unsatisfactory. We will comment upon this later on (§ 4F).

2C. Tekse (1967, p. 194) finds the Hadwiger function of (2.1) wholly unsuitable to represent his Hungarian fertility data, and prefers the γ-function in a form like

(2.6) $\gamma_T(x) = a(x - 14)^b \exp\{-c(x - 14)\}$.

As before x represents age attained, and a, b, and c are parameters to be fitted. (The subtraction of 14 from x in (2.6) corresponds to moving the origin up to age 14.)

Tekse "normalizes" his rates to $f_x^* = \hat{f}_x/\hat{R}_0$, sets $a = c^{b+1}/\Gamma(b+1)$, and estimates b and c by the method of moments and by the maximum likelihood method. He also lists an example where he claims to have used the method of least squares (Tekse, 1967, Table 1), but as the simple sum of squares of the deviations of his least squares fit exceeds the corresponding sums of squares when he applies the two other methods to the same data, we do not understand what his "least squares" procedure actually involves. As will be seen below, we have calculated several least squares estimates by his data, and have get quite different results.

Tekse (1967, p. 197) suggests the hypothesis that the Hadwiger function[x] may give a better fit to the rates of a high fertility population, whereas the $\gamma$-function of (2.6) may be preferred in the case of lower fertility. He suggests as well that "at least in Europe, there may exist some connection between the level of fertility on the one hand and the relative fertility intensities of the various age groups, i.e. the form of the fertility function on the other hand." Our results seem to cast some further light on these suggestions.

Tekse also considers some further functions, which we shall leave aside here.

2D. The estimating methods described in §§ 2B and 2C are by intention convenient for use on a table-calculator as they involve only a fair amount of calculating work. With the electronic computers now available, it is however largely possible to disregard problems caused by laboursome calculations. This gives us a much wider choice of estimating procedures.

As far as we can see, no particular advantage is gained by using the method of moments fully or partially. Instead, we have chosen the method of least squares. Thus we have selected those parameter values which minimize the expression

$$(2.7)\quad F = \sum_{x=15}^{44} (\hat{f}_x - f_x)^2,$$

where $\hat{f}_x$ still represents the observed rate at age $x$, and $f_x$ represents the value at $x$ of the function to be fitted to the data.

The algorithm used to find the minimum of F is described by Nelder and Mead (1965).

We could alternatively have used the $\chi^2$ minimizing method, but rejected this because of the undesirably great weight laid upon ages with low fertility by this method.

## 3. Choice of functions to be fitted

3A. We have used three versions of the gamma function. Two of these build upon the formula

$$(3.1)\quad \gamma(x) = a(x+d)^b \exp\{-c(x+d)\}.$$

and the third version is given by (2.5). ($x$ is still age attained, and $a$, $b$, $c$, and $d$ are parameters as before.)

x) Or alternatively a function due to Mazur.

The three versions are:

<u>Version 1</u>: $\gamma(x)$ with a,b,c, and d estimated by the least squares method (LSM),

<u>Version 2</u>: $\gamma(x)$ with a,b, and c estimated by the LSM, while d was fixed at -14, and

<u>Version 3</u>: $\gamma_Y(x)$ with a,b,c,and d estimated by the LSM.

We observe that version 2 is that of Tekse (1967) given by (2.6). Versions 1 and 3 cannot be transformed into each other by simple parameter transformations. To obtain $\gamma(x)>0$ for all $x \geq 15$, we shall require that $d>-15$ everywhere.

<u>3B.</u> A simple extension of the function given by (2.1) results from adding a new parameter d to x. This gives

$$(3.2) \quad h(x) = \frac{ab}{c\sqrt{\pi}}\left(\frac{c}{x+d}\right)^{\frac{3}{2}} \exp\{-b^2\left(\frac{c}{x+d} + \frac{x+d}{c} - 2\right)\}.$$

We have used two versions of this generalized Hadwiger function, viz.

<u>Version 1</u>: h(x) with a,b,c, and d estimated by the LSM,

and

<u>Version 2</u>: h(x) with b estimated by the LSM, d fixed at 0 and a and c estimated by (2.2).

<u>3C.</u> The last function we have given examples for, is a fourth degree polynomial of the form:

$$(3.3) \quad p(x) = a+b\ 10^{-1}(x-14) + c\ 10^{-2}(x-14)^2 + d\ 10^{-4}(x-14)^3 + e\ 10^{-6}(x-14)^4.$$

In this context p(x) has the disadvantage of giving negative values for certain combinations of x and the parameters a, b, c, d, and e. Again we require that $d>-15$ everywhere.

<u>3D.</u> We have also made some trial calculations on the beta function (Pearson I) and the Pearson IV function. Neither of these gave any promising results, so we left them aside.

## <u>4. Application to our data</u>

<u>4A.</u> We have fitted the functions given in § 3 to data from five different sets of fertility rates, viz. 1961 data for Hungary (Tekse, 1967, p. 194) and 1966 data for Norway and for the three Norwegian municipalities of Oslo, Stavanger, and Tromsø. In all cases the rates given are age-specific female fertility rates for offspring of both sexes.

The observed gross fertility rates for the five populations have been listed in column 2 in tables 3 to 7. In columns 3 to 8, we have given the fitted function values. Below these columns, we have listed the parameter values obtained as well as the corresponding least sums of squares of deviations, i.e. the minimum of F in (2.7).

The observed fertility rates and the fitted Hadwiger curves have been drawn in figures 1 to 5. Some of our findings have been summarized in table 1.

In table 2 we have listed the size of the five female populations in question. We see that they vary greatly and that the fit increases strongly with the size of the population. We also see that the fertility varies appreciably both as to fertility level and as to skew of the observed fertility curve.

(We have also made calculations on Swiss data for 1937 given by Hadwiger (1941, p. 33). This gave nothing new, however, and we have not included the results here.)

4B. The tables show that the two best alternatives are either version 1 of the gamma function or version 1 of the Hadwiger function. For the data from Hungary and Tromsø, the Hadwiger function is best, while the gamma function is best for the data from Stavanger, Oslo and Norway. The difference is small in all cases.

Thus while the second Hadwiger function gives a bad-fit to the Hungarian data, as observed by Tekse (1967) already, version 1 actually gives a very good fit.

We cannot find any connection between the fertility level, as measured by $\hat{R}_0$, and the choice between the two best functions. On the other hand the form of the fertility curve seems to have some effect, as hypothesised by Tekse (cf. § 2C above). It is notable that the fertility curves for the populations of Hungary and Tromsø are skewer than for the others.

4C. The parameter $\hat{d}$ appears as a measure of the skew of the gamma function. In our data, $\hat{d}$ in version 1 of this function varies in the range between a value slightly exceeding -15 (for Hungary) and a value of -11.13 (for Oslo). It is close to its lower bound of -15 in the populations with a skew fertility curve.

4D. Similarly $\hat{d}$ is a measure of the skew of the Hadwiger function in (3.2). (Cf. table 1 and the figures.)

On comparison of the two versions of the Hadwiger function the examples make it quite clear that the effect of letting d vary is very strong. We see that $\hat{d}$ in version 1 lies in the interval between -13.048 for Hungary and 0.101 for Oslo. The corresponding differences between the "F.min."-values of versions 1 and 2 vary from 2530.4 per cent to 7.8 per cent of "F.min." if version 1.

Figures 1 to 5 show the two versions of the Hadwiger function together with the observed values. We see that the more skew the fertility curve is, the better are the relative merits of version 1. On the other hand, the right-hand tail of version 1 can be quite "heavy". This is presumably the price we have to pay for the better smoothing at the lower ages.

4E. In table 1 we have listed the values of $\hat{c}-\hat{d}$ and $\hat{T}$ in the two versions of the Hadwiger function. It will be seen that $\hat{c}-\hat{d}$ and $\hat{T}$ are of the same size order. We have not been able to explain why this should be so.

From tables 3 to 7 we see that corresponding $\hat{a}$ of the two versions of the Hadwiger function are not much different, while the $\hat{b}$ of version 1 can be much smaller than the corresponding $\hat{b}$ of version 2, particularly for skew fertility curves. In the Norwegian data, formula (2,4) tends to overestimate the corresponding least squares estimate of b by 5 to 10 per cent of the latter. For the Hungarian data, the difference is negligible.

4F. We see that version 3 of the gamma function is a somewhat unfortunate alternative. We have included it for comparison only.

4G. The polynomial has the undesirable property of giving negative values for the age 15 and in some cases for the age 44. This can of course be corrected for by replacing all negative values by zero.

The fourth degree polynomial does not give the best fit in any of our examples. Even so, the fit is quite good for some data sets.

## 5. Conclusions

On the basis of the investigations reported above, we draw the following conclusions.

(i) The inclusion of the age-correcting parameter d in (3.1) and (3.2) may lead to a substantial increase in the fit, as measured by the least sum of squares of deviations.

(ii) On the basis of our results, it is difficult to choose between the best versions of the gamma and the Hadwiger functions. It seems possible that the latter may be the better for skew fertility curves while the former may be preferred for more symmetric curves. In any case the difference in fit seems small.

(iii) If one prefers a simple functional form rather than the very best fit, a polymonial of at least fourth degree is a reasonable choice.

## 6. Tables and figures

Table 1

| | Least sum of squares of deviation, mult. by $10^6$ | | Gross fertility rate, | Hadwiger function | | |
|---|---|---|---|---|---|---|
| | | | | Version 1 | | Version 2 |
| | Gamma Version 1 | Hadwiger Version 1 | $\hat{R}_0$ | $\hat{d}$ | $\hat{c}-\hat{d}$ | $\hat{T}$ |
| Norway ............ | 1 183 | 1 468 | 2.83 | -9.3 | 27.5 | 26.7 |
| Oslo .............. | 2 273 | 2 324 | 2.00 | 0.1 | 26.7 | 26.5 |
| Stavanger ......... | 7 779 | 7 968 | 2.73 | -2.8 | 27.3 | 26.9 |
| Tromsø ............ | 25 856 | 25 333 | 3.55 | -12.5 | 28.7 | 26.5 |
| Hungary ........... | 668 | 168 | 1.92 | -13.0 | 25.7 | 25.1 |

Table 2

| Region | Date | Number of women in the age interval 15-44 |
|---|---|---|
| Norway ......................... | 31 XII 1965 | 712 500 |
| Oslo ........................... | " | 98 992 |
| Stavanger ...................... | " | 15 866 |
| Tromsø ......................... | " | 7 021 |
| Hungary ........................ | 1 VII 1961 | 2 142 000 |

Sources: Norwegian data: NOS A 222.

Hungarian data: U.N. Demographic Yearbook, 1962, pp. 176-7.

Table 3: Norway 1966.

| Age | Fertility | Gamma functions | | | Hadwiger functions | | Polynomial |
|---|---|---|---|---|---|---|---|
| | | Version 1 | Version 2 | Version 3 | Version 1 | Version 2 | |
| 15 ....... | .00359 | 0.00276 | 0.00476 | 0.04230 | 0.00875 | 0.02330 | -0.01413 |
| 16 ....... | .01619 | 0.02069 | 0.02324 | 0.05634 | 0.02519 | 0.03928 | 0.02979 |
| 17 ....... | .04723 | 0.05086 | 0.05236 | 0.07261 | 0.05128 | 0.05970 | 0.06719 |
| 18 ....... | .08989 | 0.08576 | 0.08597 | 0.09063 | 0.08344 | 0.08323 | 0.09847 |
| 19 ....... | .12861 | 0.11938 | 0.11868 | 0.10965 | 0.11654 | 0.10793 | 0.12402 |
| 20 ....... | .14295 | 0.14790 | 0.14681 | 0.12869 | 0.14605 | 0.13163 | 0.14424 |
| 21 ....... | .17500 | 0.16941 | 0.16837 | 0.14664 | 0.16905 | 0.15234 | 0.15952 |
| 22 ....... | .17556 | 0.18342 | 0.18268 | 0.16233 | 0.18431 | 0.16854 | 0.17027 |
| 23 ....... | .18495 | 0.19033 | 0.19001 | 0.17473 | 0.19191 | 0.17932 | 0.17686 |
| 24 ....... | .18026 | 0.19105 | 0.19114 | 0.18299 | 0.19275 | 0.18441 | 0.17968 |
| 25 ....... | .20124 | 0.18674 | 0.18715 | 0.18658 | 0.18812 | 0.18406 | 0.17913 |
| 26 ....... | .18611 | 0.17857 | 0.17920 | 0.18535 | 0.17938 | 0.17894 | 0.17558 |
| 27 ....... | .16900 | 0.16763 | 0.16841 | 0.17950 | 0.16785 | 0.16995 | 0.16941 |
| 28 ....... | .14690 | 0.15504 | 0.15577 | 0.16954 | 0.15462 | 0.15811 | 0.16100 |
| 29 ....... | .13707 | 0.14145 | 0.14210 | 0.15632 | 0.14059 | 0.14439 | 0.15072 |
| 30 ....... | .12534 | 0.12758 | 0.12810 | 0.14076 | 0.12644 | 0.12969 | 0.13894 |
| 31 ....... | .11300 | 0.11392 | 0.11426 | 0.12384 | 0.11266 | 0.11476 | 0.12603 |
| 32 ....... | .10027 | 0.10081 | 0.10098 | 0.10652 | 0.09960 | 0.10019 | 0.11237 |
| 33 ....... | .10052 | 0.08851 | 0.08850 | 0.08962 | 0.08745 | 0.08640 | 0.09830 |
| 34 ....... | .08073 | 0.07715 | 0.07699 | 0.07379 | 0.07634 | 0.07369 | 0.08421 |
| 35 ....... | .06984 | 0.06683 | 0.06652 | 0.05948 | 0.06631 | 0.06222 | 0.07043 |
| 36 ....... | .06115 | 0.05755 | 0.05713 | 0.04696 | 0.05734 | 0.05205 | 0.05733 |
| 37 ....... | .05085 | 0.04930 | 0.04879 | 0.03634 | 0.04939 | 0.04318 | 0.04527 |
| 38 ....... | .04046 | 0.04203 | 0.04146 | 0.02756 | 0.04240 | 0.03554 | 0.03459 |
| 39 ....... | .03712 | 0.03567 | 0.03507 | 0.02051 | 0.03630 | 0.02904 | 0.02564 |
| 40 ....... | .02851 | 0.03015 | 0.02953 | 0.01497 | 0.03099 | 0.02358 | 0.01877 |
| 41 ....... | .02069 | 0.02539 | 0.02477 | 0.01073 | 0.02639 | 0.01903 | 0.01432 |
| 42 ....... | .01383 | 0.02131 | 0.02070 | 0.00755 | 0.02243 | 0.01527 | 0.01264 |
| 43 ....... | .00725 | 0.01783 | 0.01724 | 0.00522 | 0.01903 | 0.01219 | 0.01405 |
| 44 ....... | .00819 | 0.01487 | 0.01431 | 0.00354 | 0.01612 | 0.00969 | 0.01890 |

Female population in age-interval 15-44: 712 500

Constants:

| | Gamma functions | | | Hadwiger functions | | Polynomial |
|---|---|---|---|---|---|---|
| | Version 1 | Version 2 | Version 3 | Version 1 | Version 2 | |
| a .................. | 0.009212 | 0.006 | 0.187 | 2.957 | 2.827 | -0.065 |
| b .................. | 2.471 | 2.687 | 81.901 | 1.764 | 2.896<br>(3.043)[1] | 0.545 |
| c .................. | 0.266 | 0.277 | 2.123 | 18.239 | 26.71 | -0.367 |
| d .................. | -14.34095 | - | -25.243 | -9.263 | - | 0.682 |
| e .................. | - | - | - | - | - | -0.107 |
| F.min. 2) .......... | 0.001183 | 0.001203 | 0.008489 | 0.001468 | 0.003432 | 0.003341 |

1) The figure in parenthesis is $\hat{b}$ calculated by (2.4).

2) Least sum of squares of deviations.

Table 4. Oslo 1966

| Age | Fertility | Gamma functions | | | Hadwiger functions | | Polynomial |
|---|---|---|---|---|---|---|---|
| | | Version 1 | Version 2 | Version 3 | Version 1 | Version 2 | |
| 15 ...... | .00503 | 0.01169 | 0.00315 | 0.02877 | 0.01664 | 0.01598 | -0.00605 |
| 16 ...... | .01912 | 0.02430 | 0.01571 | 0.03863 | 0.02818 | 0.02740 | 0.02409 |
| 17 ...... | .04187 | 0.04112 | 0.03584 | 0.05016 | 0.04301 | 0.04217 | 0.04979 |
| 18 ...... | .06857 | 0.06046 | 0.05937 | 0.06304 | 0.06020 | 0.05936 | 0.07131 |
| 19 ...... | .09560 | 0.08026 | 0.08251 | 0.07674 | 0.07833 | 0.07753 | 0.08891 |
| 20 ...... | .09293 | 0.09859 | 0.10260 | 0.09057 | 0.09582 | 0.09503 | 0.10286 |
| 21 ...... | .11483 | 0.11394 | 0.11818 | 0.10370 | 0.11119 | 0.11032 | 0.11343 |
| 22 ...... | .11374 | 0.12535 | 0.12869 | 0.11531 | 0.12328 | 0.12223 | 0.12087 |
| 23 ...... | .12235 | 0.13243 | 0.13428 | 0.12459 | 0.31141 | 0.13006 | 0.12544 |
| 24 ...... | .11570 | 0.13524 | 0.13542 | 0.13090 | 0.13534 | 0.13360 | 0.12741 |
| 25 ...... | .15724 | 0.13417 | 0.13291 | 0.13383 | 0.13525 | 0.13304 | 0.12704 |
| 26 ...... | .14035 | 0.12985 | 0.12753 | 0.13324 | 0.13161 | 0.12893 | 0.12457 |
| 27 ...... | .13179 | 0.12297 | 0.12007 | 0.12925 | 0.12508 | 0.12197 | 0.12028 |
| 28 ...... | .11666 | 0.11428 | 0.11123 | 0.12224 | 0.11640 | 0.11292 | 0.11440 |
| 29 ...... | .09127 | 0.10443 | 0.10162 | 0.11277 | 0.10632 | 0.10256 | 0.10720 |
| 30 ...... | .09455 | 0.09401 | 0.09172 | 0.10157 | 0.09548 | 0.09156 | 0.09892 |
| 31 ...... | .08049 | 0.08351 | 0.08191 | 0.08934 | 0.08447 | 0.08049 | 0.08982 |
| 32 ...... | .06923 | 0.07329 | 0.07247 | 0.07679 | 0.07370 | 0.06977 | 0.08014 |
| 33 ...... | .07020 | 0.06361 | 0.06357 | 0.06454 | 0.06352 | 0.05971 | 0.07014 |
| 34 ...... | .06056 | 0.05467 | 0.05535 | 0.05305 | 0.05413 | 0.05052 | 0.06006 |
| 35 ...... | .05328 | 0.04656 | 0.04786 | 0.04268 | 0.04566 | 0.04230 | 0.05014 |
| 36 ...... | .03916 | 0.03932 | 0.04113 | 0.03362 | 0.03815 | 0.03508 | 0.04064 |
| 37 ...... | .02879 | 0.03295 | 0.03515 | 0.02594 | 0.03161 | 0.02884 | 0.03179 |
| 38 ...... | .02261 | 0.02741 | 0.02989 | 0.01961 | 0.02598 | 0.02352 | 0.02383 |
| 39 ...... | .02611 | 0.02266 | 0.02529 | 0.01454 | 0.02120 | 0.01903 | 0.01701 |
| 40 ...... | .01760 | 0.01892 | 0.02131 | 0.01057 | 0.01719 | 0.01530 | 0.01157 |
| 41 ...... | .01143 | 0.01521 | 0.01788 | 0.00754 | 0.01385 | 0.01223 | 0.00774 |
| 42 ...... | .00410 | 0.01235 | 0.01495 | 0.00528 | 0.01109 | 0.00971 | 0.00575 |
| 43 ...... | .00220 | 0.00999 | 0.01245 | 0.00363 | 0.00884 | 0.00767 | 0.00585 |
| 44 ...... | .00589 | 0.00804 | 0.01034 | 0.00245 | 0.00701 | 0.00603 | 0.00827 |

Female population in age-interval 15-44: 98 992.

Constants:

| | Gamma functions | | | Hadwiger functions | | Polynomial |
|---|---|---|---|---|---|---|
| | Version 1 | Version 2 | Version 3 | Version 1 | Version 2 | |
| a .................... | 0.000076 | 0.004 | 0.134 | 2.069 | 1.999 | -0.041 |
| b .................... | 4.766 | 2.721 | 81.889 | 2.921 | 2.953 (3.226)[1)] | 0.373 |
| c .................... | 0.365 | 0.278 | 2.161 | 26.825 | 26.54 | -0.249 |
| d .................... | -11.13333 | - | -25.330 | 0.101 | - | 0.451 |
| e .................... | - | - | - | - | - | -0.053 |
| F.min.[1)] ............ | 0.002273 | 0.002714 | 0.003513 | 0.002324 | 0.002505 | 0.002515 |

1) See footnotes in table 1.

Table 5: Stavanger 1966

| Age | Fertility | Gamma functions Version 1 | Gamma functions Version 2 | Gamma functions Version 3 | Hadwiger functions Version 1 | Hadwiger functions Version 2 | Polynomial |
|---|---|---|---|---|---|---|---|
| 15 ........ | .00328 | 0.01181 | 0.00402 | 0.03689 | 0.01842 | 0.02028 | -0.00930 |
| 16 ........ | .02677 | 0.02822 | 0.01993 | 0.04944 | 0.03333 | 0.03473 | 0.02862 |
| 17 ........ | .03697 | 0.05088 | 0.04553 | 0.06414 | 0.05306 | 0.05353 | 0.06149 |
| 18 ........ | .07524 | 0.07713 | 0.07569 | 0.08064 | 0.07628 | 0.07556 | 0.08953 |
| 19 ........ | .14586 | 0.10395 | 0.10572 | 0.09832 | 0.10093 | 0.09908 | 0.11296 |
| 20 ........ | .12265 | 0.12868 | 0.13223 | 0.11636 | 0.12476 | 0.12206 | 0.13201 |
| 21 ........ | .15065 | 0.14937 | 0.15339 | 0.13377 | 0.14573 | 0.14257 | 0.14692 |
| 22 ........ | .17359 | 0.16487 | 0.16825 | 0.14949 | 0.16232 | 0.15905 | 0.15793 |
| 23 ........ | .13158 | 0.17474 | 0.17688 | 0.16252 | 0.17365 | 0.17053 | 0.16531 |
| 24 ........ | .17438 | 0.17914 | 0.17982 | 0.17200 | 0.17950 | 0.17660 | 0.16930 |
| 25 ........ | .17843 | 0.17859 | 0.17792 | 0.17732 | 0.18013 | 0.17742 | 0.17019 |
| 26 ........ | .15886 | 0.17388 | 0.17214 | 0.17817 | 0.17618 | 0.17353 | 0.16825 |
| 27 ........ | .17687 | 0.16588 | 0.16345 | 0.17466 | 0.16851 | 0.16575 | 0.16377 |
| 28 ........ | .16814 | 0.15548 | 0.15274 | 0.16707 | 0.15804 | 0.15501 | 0.15704 |
| 29 ........ | .18537 | 0.14349 | 0.14077 | 0.15606 | 0.14568 | 0.14225 | 0.14837 |
| 30 ........ | .13221 | 0.13062 | 0.12818 | 0.14243 | 0.13226 | 0.12836 | 0.13807 |
| 31 ........ | .11594 | 0.11746 | 0.11550 | 0.12708 | 0.11845 | 0.11408 | 0.12646 |
| 32 ........ | .09068 | 0.10448 | 0.10311 | 0.11090 | 0.10480 | 0.10000 | 0.11386 |
| 33 ........ | .08768 | 0.09202 | 0.09128 | 0.09470 | 0.09172 | 0.08657 | 0.10062 |
| 34 ........ | .07765 | 0.08033 | 0.08020 | 0.07917 | 0.07950 | 0.07411 | 0.08707 |
| 35 ........ | .06550 | 0.06955 | 0.06999 | 0.06484 | 0.06830 | 0.06279 | 0.07357 |
| 36 ........ | .07362 | 0.05978 | 0.06071 | 0.05204 | 0.05821 | 0.05270 | 0.06047 |
| 37 ........ | .03299 | 0.05103 | 0.05237 | 0.04094 | 0.04925 | 0.04385 | 0.04816 |
| 38 ........ | .02796 | 0.04329 | 0.04495 | 0.03159 | 0.04140 | 0.03620 | 0.03699 |
| 39 ........ | .05081 | 0.03651 | 0.03839 | 0.02392 | 0.03460 | 0.02967 | 0.02737 |
| 40 ........ | .03018 | 0.03062 | 0.03265 | 0.01778 | 0.02875 | 0.02416 | 0.01968 |
| 41 ........ | .02994 | 0.02556 | 0.02766 | 0.01297 | 0.02377 | 0.01955 | 0.01432 |
| 42 ........ | .01255 | 0.02124 | 0.02335 | 0.00930 | 0.01956 | 0.01572 | 0.01170 |
| 43 ........ | .00729 | 0.01757 | 0.01964 | 0.00655 | 0.01603 | 0.01258 | 0.01224 |
| 44 ........ | .00469 | 0.01447 | 0.01646 | 0.00454 | 0.01309 | 0.01002 | 0.01636 |

Female population in age-interval 15-44: 15 866

Constants:

| | Gamma functions Version 1 | Gamma functions Version 2 | Gamma functions Version 3 | Hadwiger functions Version 1 | Hadwiger functions Version 2 | Polynomial |
|---|---|---|---|---|---|---|
| a .................... | 0.000570 | 0.005 | 0.178 | 2.843 | 2.732 | -0.052 |
| b .................... | 3.844 | 2.697 | 81.728 | 2.533 | 2.907 (3.076)[1] | 0.459 |
| c .................... | 0.317 | 0.268 | 2.054 | 24.577 | 26.94 | -0.272 |
| d .................... | -12.23783 | - | -25.694 | -2.750 | - | 0.322 |
| e .................... | - | - | - | - | - | 0.339 |
| F.min.[1] ............ | 0.007779 | 0.008140 | 0.010185 | 0.007968 | 0.008392 | 0.007932 |

1) See footnotes in table 1.

Table 8: Tromsø 1966

| Age | Fertility | Gamma functions | | | Hadwiger functions | | Polynomial |
|---|---|---|---|---|---|---|---|
| | | Version 1 | Version 2 | Version 3 | Version 1 | Version 2 | |
| 15 ........ | .01132 | 0.00000 | 0.01926 | 0.07959 | 0.00495 | 0.05038 | -0.01047 |
| 16 ........ | .04264 | 0.04447 | 0.05902 | 0.09753 | 0.03480 | 0.07421 | 0.05731 |
| 17 ........ | .07744 | 0.09977 | 0.10357 | 0.11660 | 0.08817 | 0.10081 | 0.11097 |
| 18 ........ | .12635 | 0.14752 | 0.14467 | 0.13606 | 0.14435 | 0.12808 | 0.15218 |
| 19 ........ | .21122 | 0.18392 | 0.17835 | 0.15510 | 0.18919 | 0.15383 | 0.18250 |
| 20 ........ | .18950 | 0.20883 | 0.20315 | 0.17278 | 0.21849 | 0.17617 | 0.20339 |
| 21 ........ | .32051 | 0.22350 | 0.21911 | 0.18825 | 0.23347 | 0.19375 | 0.21623 |
| 22 ........ | .21515 | 0.22963 | 0.22707 | 0.20068 | 0.23723 | 0.20577 | 0.22229 |
| 23 ........ | .22491 | 0.22898 | 0.22825 | 0.20944 | 0.23299 | 0.21204 | 0.22274 |
| 24 ........ | .16892 | 0.22318 | 0.22398 | 0.21412 | 0.22346 | 0.21283 | 0.21867 |
| 25 ........ | .20385 | 0.21365 | 0.21556 | 0.21450 | 0.21072 | 0.20877 | 0.21105 |
| 26 ........ | .17992 | 0.20157 | 0.20414 | 0.21072 | 0.19626 | 0.20067 | 0.20078 |
| 27 ........ | .19397 | 0.18791 | 0.19074 | 0.20304 | 0.18112 | 0.18946 | 0.18864 |
| 28 ........ | .13596 | 0.17340 | 0.17618 | 0.19202 | 0.16600 | 0.17605 | 0.17534 |
| 29 ........ | .20207 | 0.15865 | 0.16112 | 0.17830 | 0.15135 | 0.16128 | 0.16146 |
| 30 ........ | .12255 | 0.14408 | 0.14609 | 0.16260 | 0.13745 | 0.14588 | 0.14750 |
| 31 ........ | .16854 | 0.13001 | 0.13146 | 0.14572 | 0.12444 | 0.13045 | 0.13388 |
| 32 ........ | .08556 | 0.11665 | 0.11750 | 0.12836 | 0.11241 | 0.11545 | 0.12090 |
| 33 ........ | .12025 | 0.10413 | 0.10440 | 0.11121 | 0.10135 | 0.10122 | 0.10877 |
| 34 ........ | .13366 | 0.09255 | 0.09226 | 0.09478 | 0.09126 | 0.08800 | 0.09760 |
| 35 ........ | .09783 | 0.08192 | 0.08114 | 0.07948 | 0.08209 | 0.07592 | 0.08742 |
| 36 ........ | .09341 | 0.07225 | 0.07105 | 0.06562 | 0.07379 | 0.06504 | 0.07815 |
| 37 ........ | .04167 | 0.06351 | 0.06196 | 0.05335 | 0.06629 | 0.05536 | 0.06960 |
| 38 ........ | .06030 | 0.05567 | 0.05384 | 0.04273 | 0.05953 | 0.04685 | 0.06151 |
| 39 ........ | .06034 | 0.04866 | 0.04652 | 0.03372 | 0.05344 | 0.03943 | 0.05351 |
| 40 ........ | .03766 | 0.04242 | 0.04025 | 0.02623 | 0.04797 | 0.03302 | 0.04514 |
| 41 ........ | .02463 | 0.03690 | 0.03465 | 0.02012 | 0.04306 | 0.02753 | 0.03583 |
| 42 ........ | .01500 | 0.03203 | 0.02975 | 0.01522 | 0.03865 | 0.02285 | 0.02493 |
| 43 ........ | .00901 | 0.02774 | 0.02548 | 0.01136 | 0.03470 | 0.01889 | 0.01169 |
| 44 ........ | .01081 | 0.02399 | 0.02177 | 0.00836 | 0.03115 | 0.01556 | -0.00476 |

Female population in age-interval 15-44: 7 021

Constants:

| | Gamma functions | | | Hadwiger functions | | Polynomial |
|---|---|---|---|---|---|---|
| | Version 1 | Version 2 | Version 3 | Version 1 | Version 2 | |
| a .................... | 0.054116 | 0.024 | 0.215 | 3.891 | 3.553 | -0.094 |
| b .................... | 1.449 | 1.987 | 82.516 | 1.144 | 2.586 (2.707)[1] | 0.922 |
| c .................... | 0.196 | 0.223 | 1.641 | 16.092 | 26.45 | -0.886 |
| d .................... | -14.99999 | - | -24.593 | -12.585 | - | 3.168 |
| e .................... | - | - | - | - | - | -4.022 |
| F.min.[1] ............ | 0.025856 | 0.027162 | 0.045543 | 0.025333 | 0.036178 | 0.026822 |

1) See footnotes in table 1.

Table 7: Hungary 1961

| Age | Fertility | Gamma functions | | | Hadwiger functions | | Polynomial |
|---|---|---|---|---|---|---|---|
| | | Version 1 | Version 2 | Version 3 | Version 1 | Version 2 | |
| 15 ....... | .00410 | 0.00000 | 0.00418 | 0.03313 | 0.00032 | 0.02124 | -0.02608 |
| 16 ....... | .01430 | 0.01459 | 0.02275 | 0.04733 | 0.00920 | 0.03583 | 0.02748 |
| 17 ....... | .03980 | 0.04880 | 0.05268 | 0.06433 | 0.03976 | 0.05406 | 0.06914 |
| 18 ....... | .07760 | 0.08672 | 0.08599 | 0.08328 | 0.08258 | 0.07437 | 0.10034 |
| 19 ....... | .12290 | 0.11897 | 0.11589 | 0.10285 | 0.12122 | 0.09466 | 0.12244 |
| 20 ....... | .14660 | 0.14164 | 0.13836 | 0.12129 | 0.14719 | 0.11283 | 0.13671 |
| 21 ....... | .16350 | 0.15415 | 0.15193 | 0.13677 | 0.15955 | 0.12714 | 0.14436 |
| 22 ....... | .16120 | 0.15769 | 0.15692 | 0.14761 | 0.16085 | 0.13650 | 0.14650 |
| 23 ....... | .15400 | 0.15417 | 0.15467 | 0.15265 | 0.15450 | 0.14054 | 0.14415 |
| 24 ....... | .14420 | 0.14560 | 0.14693 | 0.15138 | 0.14349 | 0.13951 | 0.13826 |
| 25 ....... | .12870 | 0.13380 | 0.13548 | 0.14418 | 0.13009 | 0.13411 | 0.12969 |
| 26 ....... | .11320 | 0.12027 | 0.12138 | 0.13198 | 0.11589 | 0.12534 | 0.11923 |
| 27 ....... | .10160 | 0.10616 | 0.10740 | 0.11621 | 0.10190 | 0.11424 | 0.10756 |
| 28 ....... | .08660 | 0.09228 | 0.09298 | 0.09852 | 0.08871 | 0.10184 | 0.09531 |
| 29 ....... | .07530 | 0.07917 | 0.07929 | 0.08050 | 0.07665 | 0.08899 | 0.08301 |
| 30 ....... | .06650 | 0.06717 | 0.06672 | 0.06343 | 0.06583 | 0.07639 | 0.07111 |
| 31 ....... | .05660 | 0.05643 | 0.05550 | 0.04825 | 0.05629 | 0.06452 | 0.05997 |
| 32 ....... | .04890 | 0.04700 | 0.04569 | 0.03545 | 0.04795 | 0.05372 | 0.04987 |
| 33 ....... | .04290 | 0.03885 | 0.03727 | 0.02518 | 0.04073 | 0.04414 | 0.04101 |
| 34 ....... | .03650 | 0.03190 | 0.03016 | 0.01730 | 0.03453 | 0.03584 | 0.03351 |
| 35 ....... | .03170 | 0.02603 | 0.02422 | 0.01151 | 0.02921 | 0.02878 | 0.02742 |
| 36 ....... | .02530 | 0.02112 | 0.01932 | 0.00742 | 0.02468 | 0.02289 | 0.02266 |
| 37 ....... | .02440 | 0.01706 | 0.01532 | 0.00464 | 0.02083 | 0.01804 | 0.01912 |
| 38 ....... | .01840 | 0.01371 | 0.01208 | 0.00281 | 0.01756 | 0.01409 | 0.01658 |
| 39 ....... | .01540 | 0.01097 | 0.00947 | 0.00165 | 0.01480 | 0.01093 | 0.01475 |
| 40 ....... | .01100 | 0.00875 | 0.00739 | 0.00095 | 0.01247 | 0.00842 | 0.01323 |
| 41 ....... | .00990 | 0.00695 | 0.00575 | 0.00052 | 0.01050 | 0.00644 | 0.01157 |
| 42 ....... | .00580 | 0.00551 | 0.00445 | 0.00028 | 0.00884 | 0.00489 | 0.00923 |
| 43 ....... | .00380 | 0.00435 | 0.00343 | 0.00015 | 0.00744 | 0.00370 | 0.00556 |
| 44 ....... | .00260 | 0.00343 | 0.00263 | 0.00008 | 0.00626 | 0.00278 | -0.00013 |

Female population in age-interval 15-44: 2 142 000

Constants:

| | Gamma functions | | | Hadwiger functions | | Polynomial |
|---|---|---|---|---|---|---|
| | Version 1 | Version 2 | Version 3 | Version 1 | Version 2 | |
| a ................ | 0.020013 | 0.006 | 0.153 | 1.963 | 1.923 | -0.093 |
| b ................ | 2.198 | 2.969 | 92.170 | 1.373 | 3.081 (3.068)[1] | 0.742 |
| c ................ | 0.316 | 0.304 | 3.398 | 12.647 | 25.19 | -0.752 |
| d ................ | -15.00000[2] | - | -23.303 | -13.048 | - | 2.751 |
| e ................ | - | - | - | - | - | -3.454 |
| F.min.[1] .......... | 0.000668 | 0.001161 | 0.008269 | 0.000168 | 0.006099 | 0.003616 |

1) See footnotes in table 1.
2) This figure is only slightly larger than -15.

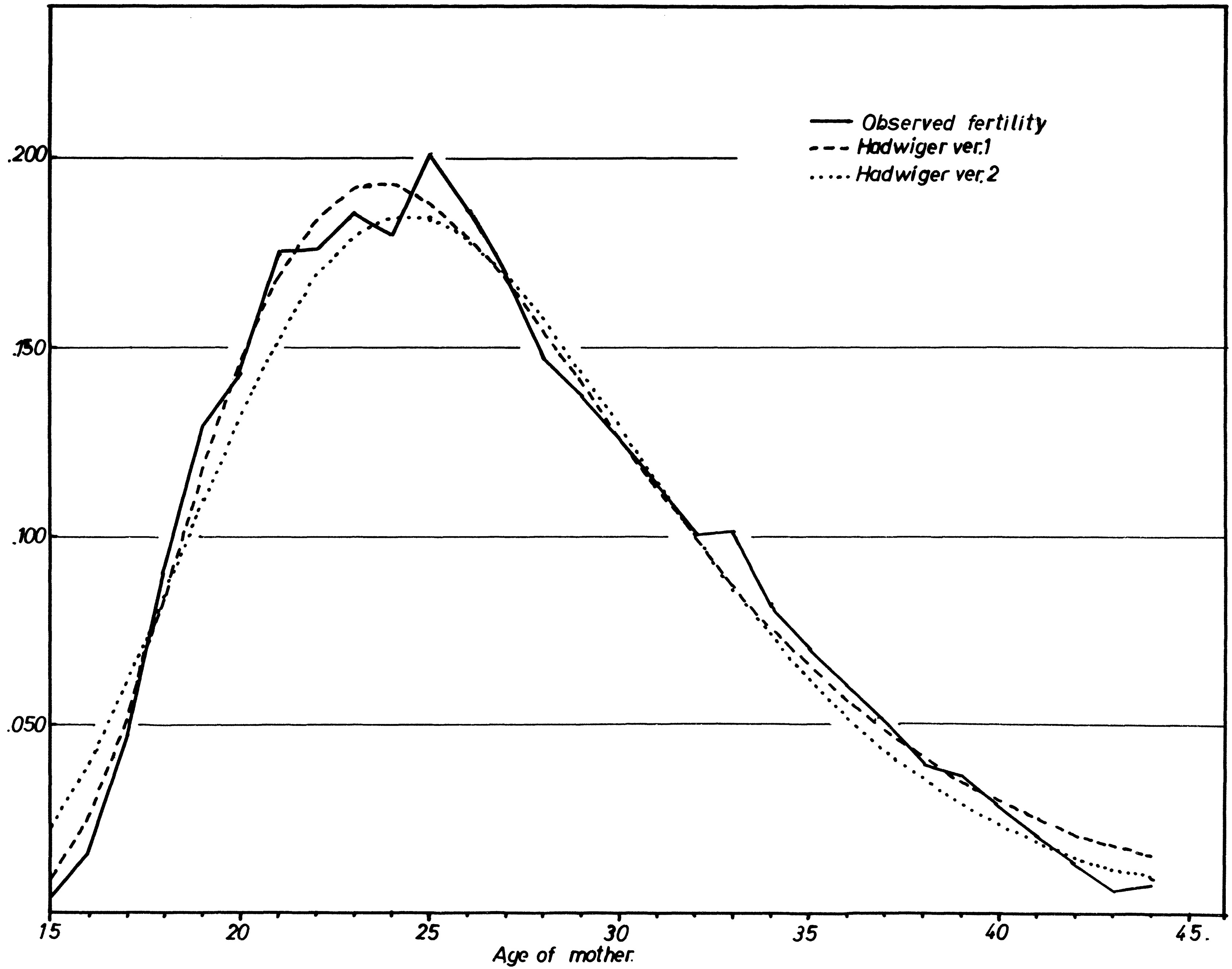



Fig. 1. Observed and smoothed fertility. Norway, 1966.

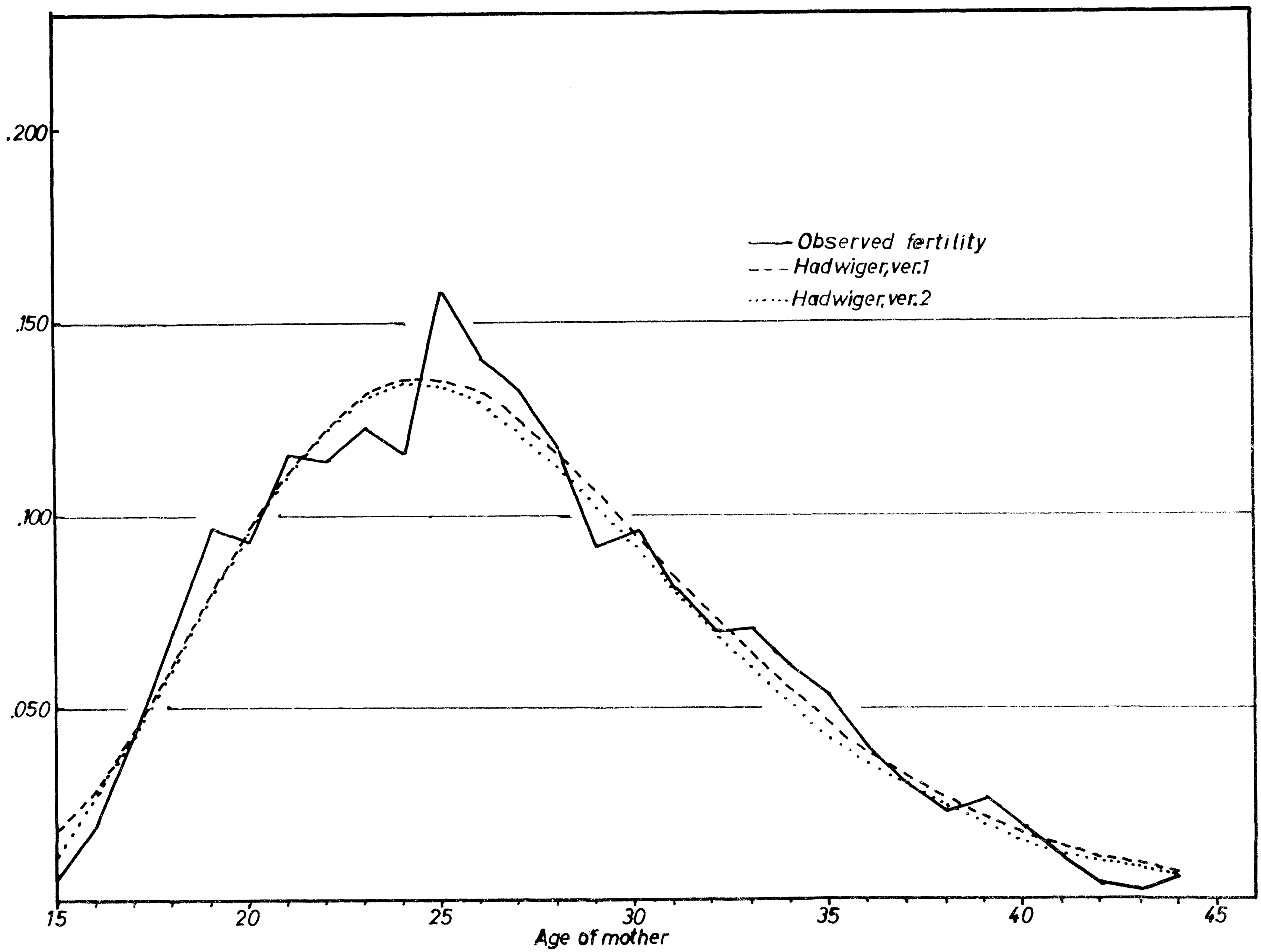



Fig. 2. Observed and smoothed fertility. Oslo, 1966.

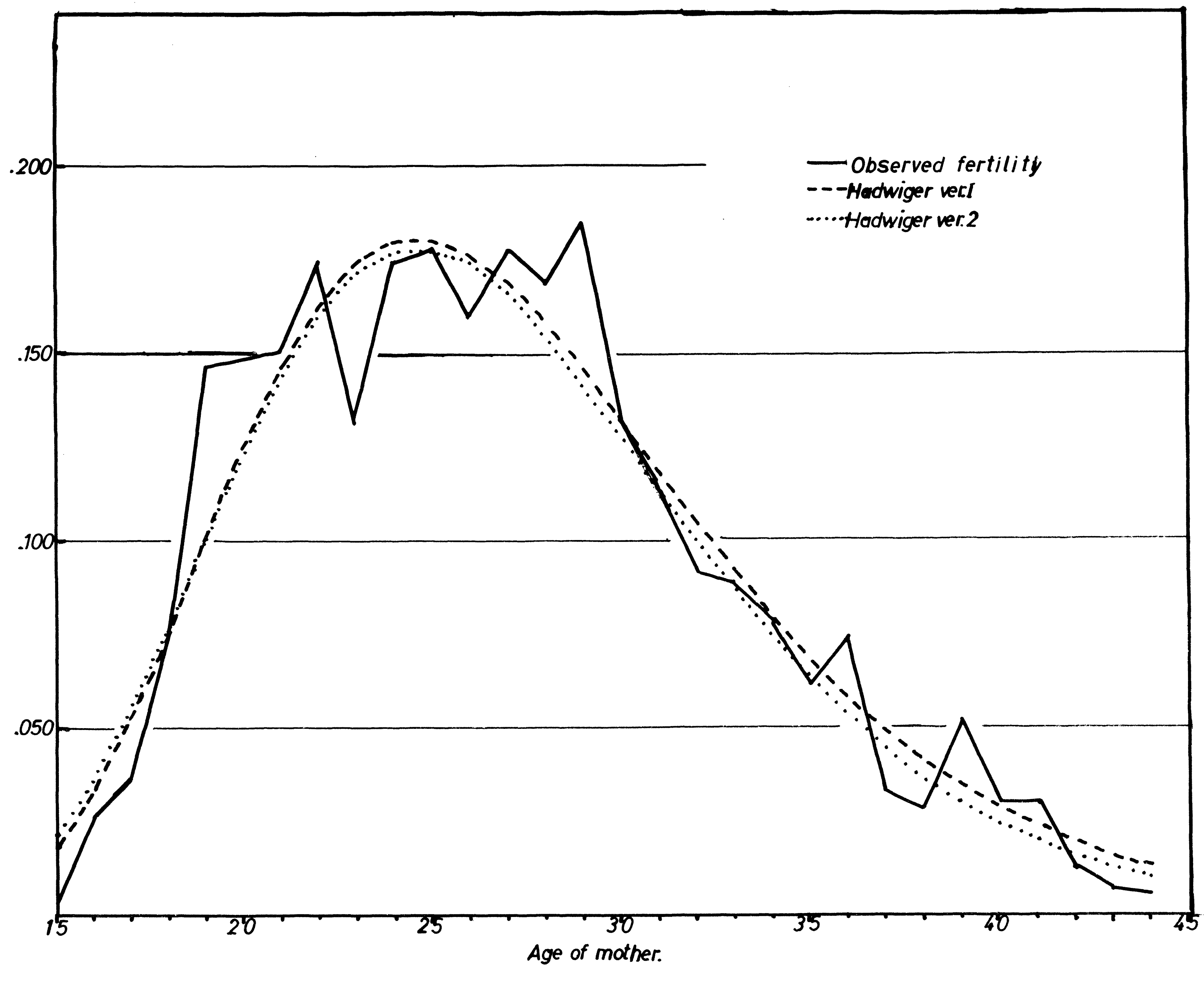
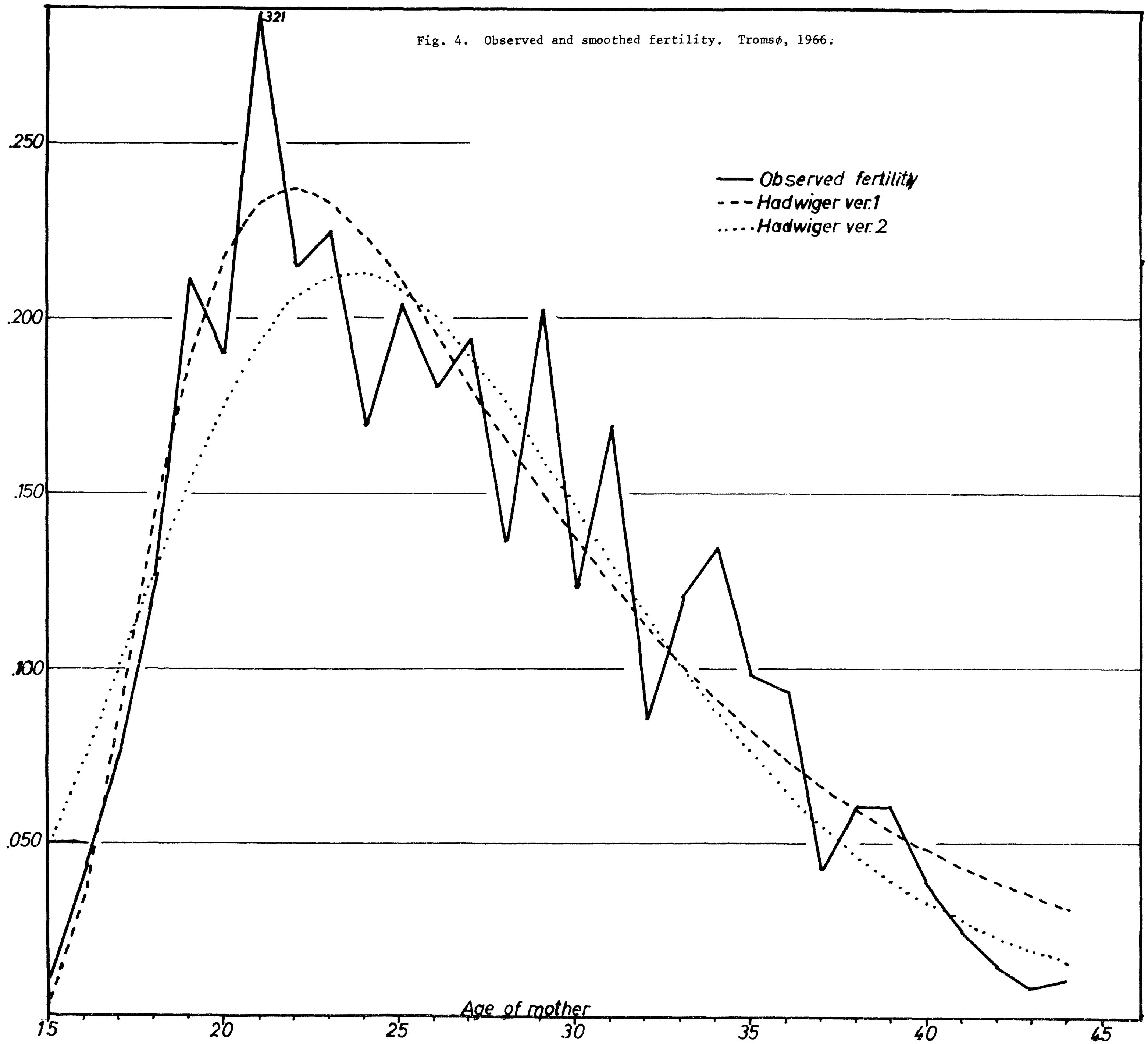



Fig. 3. Observed and smoothed fertility. Stavanger 1966.

Fig. 4. Observed and smoothed fertility. Tromsø, 1966.



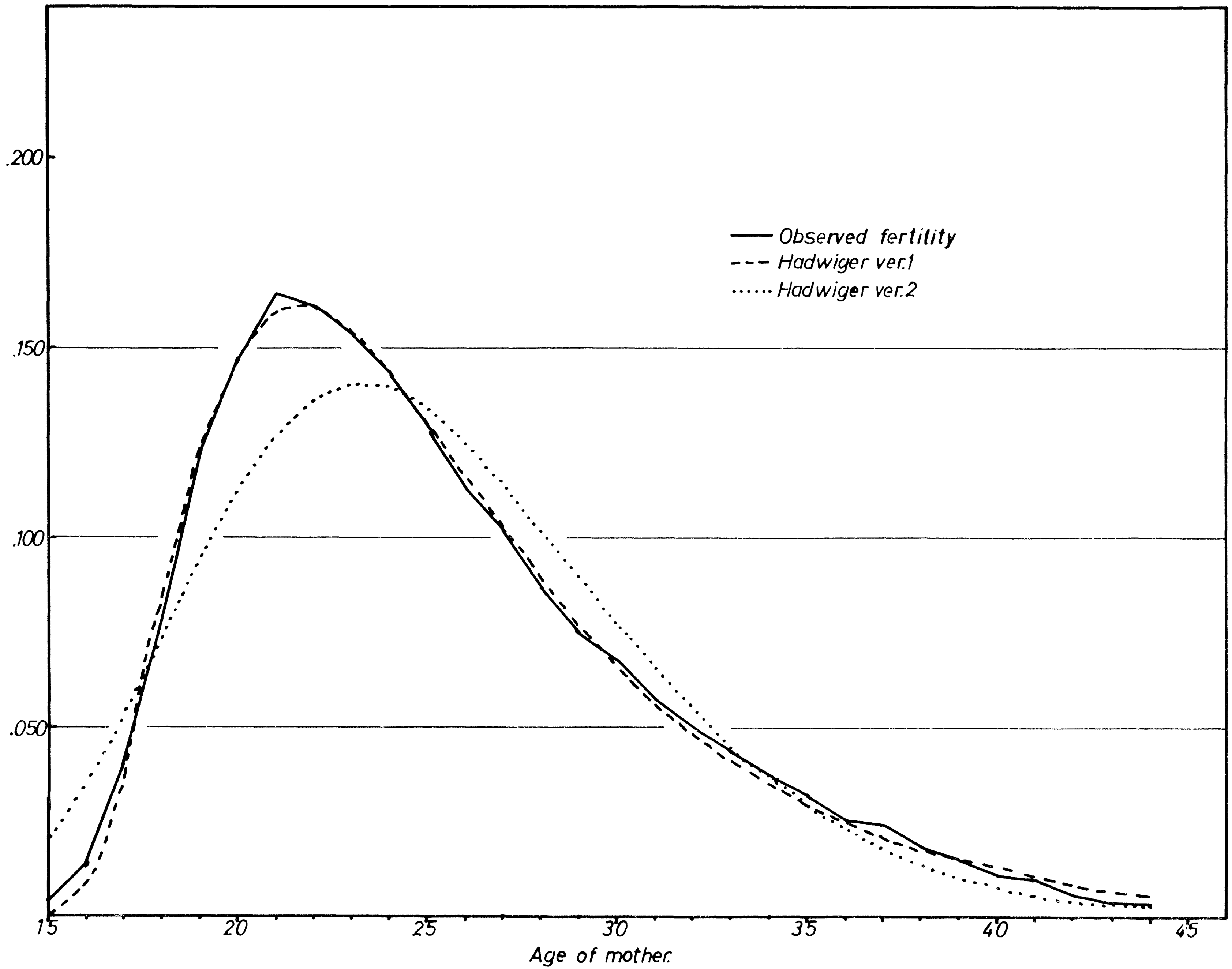



Fig. 5. Observed and smoothed fertility. Hungary, 1961.

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