Discussion Paper

Central Bureau of Statistics, P.B. 8131 Dep, 0033 Oslo 1, Norway

No. 5 25 March 1985

CERTAINTY EQUIVALENCE PROCEDURES IN DECISION-MAKING UNDER UNCERTAINTY: AN EMPIRICAL APPLICATION

BY

IULIE ASLAKSEN AND OLAV BJERKHOLT

CONTENTS

ABSTRACT

The paper outlines an approach to optimal planning procedures in a national economy with petroleum resources and other assets with stochastic rates of return. The starting point is an adaptation of some ideas put forward by Leif Johansen on parametric certainty equivalence procedures. These are used both in solving the dynamic stochastic optimization model and in the derivation of revealed preferences from recent projections of the Norwegian economy.

1 The Johansen approach to certainty equivalence procedures in decision-making under uncertainty*

The application of certainty equivalence procedures is a useful method for simplifying a decision problem involving uncertainty. In the theory of economic policy and planning certainty equivalence procedures have mainly been elaborated in the case of a quadratic objective function combined with a linear structural model. One of the many contributions of Leif Johansen in this field is the generalization of the usual certainty equivalence procedure to the case of an objective function expressed in terms of combinations of exponential functions, by the so-called "parametric certainty equivalence". The idea is to formulate a procedure for optimal decision-making under uncertainty which is similar to the one which would be valid in the case of full certainty by permitting an appropriate adjustment of the parameters of the objective function.

Following the notation of Johansen (1980), we consider a decision problem of the following general nature:

x = a variable or a vector of variables to which we attach preferences

a = a decision or action represented by a vector of instrument variables

A = a set of possible actions, i.e. we have $a \in A$

z = a vector of random variables

The values obtained for x are determined by the action taken and the random variables by a reduced form system which we write as

(1.1) $x = f(a,z)$

The objective function is written as

---

*) Jens Stoltenberg has given very valuable assistance, in particular with the calculations.

$$(1.2) \quad U = U(x;\alpha)$$

where α is a vector of parameters characterizing the objective function, i.a. the degree of risk aversion.

The optimization problem is to choose a decision in order to maximize the expected value of U. In making the decision we do not know the actual values of the random variables z, but only the probability distribution of z. The optimal decision is determined by

$$(1.3) \quad \max_{a\in A} E(U(x;\alpha)) = \max_{a\in A} E(U(f(a,z);\alpha))$$

A certainty equivalence procedure for solving the problem (1.3) will consist in replacing the stochastic variables by non-stochastic values $\tilde{z}$ and then solving the problem

$$(1.4) \quad \max_{a\in A} U(f(a,\tilde{z});\alpha)$$

Such a procedure is often followed in practice, for instance with $\tilde{z} = Ez$. More generally the certainty equivalent $\tilde{z}$ may be considered as derived from the probability distribution of z.

Apart from the wellknown linear-quadratic case it is not easy to obtain certainty equivalence procedures for which (1.4) yields the same solution as (1.3). However, one may obtain "parametric certainty equivalence", where the parameters of the objective function are adjusted in a prescribed way in order to take uncertainty into account.

The parametric certainty equivalence procedure can be formulated as

$$(1.5) \quad \max_{a\in A} U(f(a,\tilde{z});\tilde{\alpha})$$

where $\tilde{\alpha}$ is a modified parameter vector, derived from the original parameter vector α and the probability distribution of the stochastic variables z. As above, $\tilde{z}$ is a non-stochastic value representing z, for instance Ez.

The parametric certainty equivalence procedure as formulated by (1.5) involves no more complexity than what originates from the form of the objective function U, the structural form f and the form of the feasible set A, i.e. those elements which are present also in the case of certainty. The usefulness of the certainty equivalence procedure depends on whether or not we can establish a sufficiently simple transformation to derive the modified parameter vector $\bar{\bar{\alpha}}$.

In the following such a procedure will be derived for the case of an objective function expressed in terms of exponential functions when the probability distribution is restricted to the class of multi-normal distributions.

Although the procedure in the case of parametric certainty equivalence is the same as under certainty, once the modified parameter vector $\bar{\bar{\alpha}}$ has been established, the decisions actually taken will now in general be different under uncertainty than under certainty, because the parameter value $\bar{\bar{\alpha}}$ will depend on the probability distribution which implies that the existence of uncertainty makes us change our decision as compared to what we would do in the absence of uncertainty.

For a scalar x the exponential objective function is

$$(1.6) \qquad U(x) = -B\exp(-\beta x) \qquad\qquad B>0,\ \beta>0$$

This form implies constant absolute risk aversion:

$$(1.7) \qquad -U''(x)/U'(x) = \beta = \text{coefficient of absolute risk aversion}$$

For a vector of variables $x = (x_1,\ldots,x_n)$ a sum of exponential functions will yield the objective function

$$(1.8) \qquad U(x) = -\Sigma\, B_i\exp(-\beta_i x_i) \qquad\qquad B_i>0,\ \beta_i>0$$

The certainty equivalence procedure is derived from the following observation:

If x is a stochastic variable, normally distributed with expected value Ex and standard deviation σ_x, then the expected value of the objective function (1.6) can be expressed as

$$(1.9) \qquad EU(x) \;=\; -B\exp(-\beta\tilde{x})$$

where $\tilde{x} = Ex - 1/2\ \beta\ \sigma_x^2$

The certainty equivalent of the stochastic variable x is the expected value minus a correction term which is proportional to the variance and the risk aversion coefficient.

The expected value of (1.6) can also be written as

$$(1.10) \qquad EU(x) \;=\; -\tilde{B}\exp(-\beta\ Ex)$$

where $\tilde{B} \;=\; B\exp(1/2\ \beta^2\ \sigma_x^2)$

This formulation is the link between the general parametric certainty equivalence approach and the sum-of-exponentials objective functions. The parameter vector α of (1.8) is

$$\alpha \;=\; (B_1, \ldots, B_n, \beta_1, \ldots, \beta_n)$$

If $x_1, \ldots, x_n$ are normally distributed with expected values Ex_i and standard deviations σ_i, the expected value of (1.8) can - using (1.10) - be written as

$$(1.11) \qquad EU(x;\alpha) \;=\; U(Ex;\tilde{\alpha})$$

where the modified parameter vector $\tilde{\alpha}$ is given by

$$(1.12) \qquad \tilde{\alpha} \;=\; (\tilde{B}_1, \ldots, \tilde{B}_n, \beta_1, \ldots, \beta_n)$$

$$= (B_1\exp(1/2\ \beta_1^2\sigma_1^2), \ldots, B_n\exp(1/2\ \beta_n^2\ \sigma_n^2), \beta_1, \ldots, \beta_n)$$

The modification of the parameters involved in this procedure affects only $B_1, \ldots, B_n$ and not $\beta_1, \ldots, \beta_n$.

2 An illustration of the parametric certainty equivalence procedure

In order to illustrate the parametric certainty equivalence procedure, we introduce the optimization problem which will be elaborated in the following sections.

We consider the long-term macroeconomic planning problem in which a substantial part of national income is accrued from investments in uncertain sources of income. The planning problem consists in determining the optimal trade-off between consumption throughout the planning period and national wealth by the end of the planning period. The concept of terminal wealth is intended to represent future consumption possibilities beyond the given planning horizon of length T.

As an illustration of the parametric certainty equivalence procedure we first consider a static analogy to this dynamic optimization problem. We assume that preferences are formulated in terms of consumption growth over the planning period (C) and total wealth by the end of the planning period (W)

$$(2.1) \quad U(C,W) = -B\exp(-bC) - G\exp(-gW)$$

If there were no uncertainty involved, we would choose from the feasible points so as to maximize (2.1). The indifference curves of (2.1) are characterized by the marginal rate of substitution between C and W given by

$$(2.2) \quad \frac{dW}{dc} = -\frac{bB}{gG}\exp(gW - bC)$$

Given the assumption of normal probability distributions, the parametric certainty equivalence procedure entails the following transformation of (2.1)

$$(2.3) \quad \tilde{U}(EC,EW) = -\tilde{B}\exp(-b\ EC) - \tilde{G}\exp(-g\ EW)$$

$$\text{where } \tilde{B} = B\exp(1/2\ b^2\sigma_c^2)$$

$$\text{and } \tilde{G} = G\exp(1/2\ g^2\sigma_w^2)$$

The standard deviations of C and W are denoted by σ_c and σ_w respectively. The certainty equivalence procedure consists in choosing EC and EW so as to maximize (2.3). The marginal rate of substitution is now expressed as

$$(2.4) \quad \frac{dEW}{dEC} = -\frac{bB}{gG}\exp(1/2(b^2\sigma_c^2 - g^2\sigma_w^2))\exp(gEW - bEC)$$

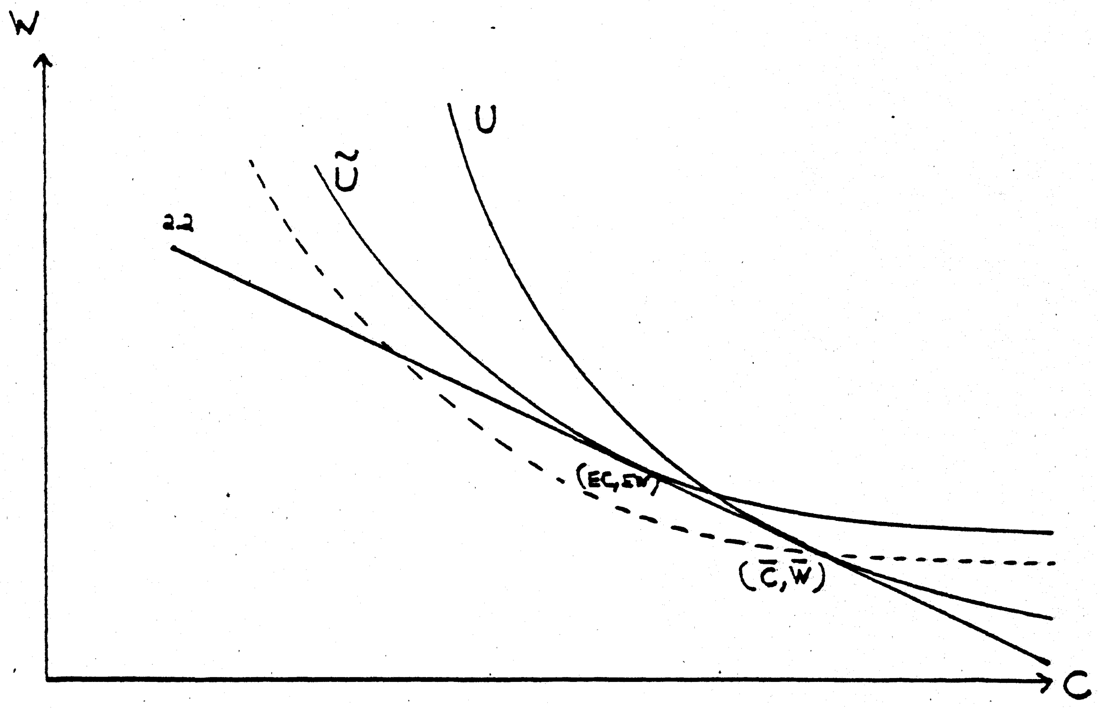
First it can be noted that uncertainty has no effect on the actual decisions in the case where

$$b\sigma_c = g\sigma_w$$

If this is not the case, the indifference curves of (2.3) will be twisted as a consequence of uncertainty. Furthermore, a partial increase in σ_w will make the indifference curve flatter while a partial increase in σ_c will make the indifference curve steeper. This will in general mean that a larger σ_c tends to induce a change in the decision in the direction of a larger value of EC, while a larger value of σ_w tends to induce a change in the decision in the direction of a larger value of EW.

In figure 1 the consequences of uncertainty are illustrated in the case where the risk adjustment term of terminal wealth is the larger, which implies a flatter indifference curve compared to the case where uncertainty is disregarded.

Figure 1. Consequences of uncertainty when $g\sigma_w > b\sigma_c$.



The indifference curve U corresponds to the case where uncertainty is disregarded, and in this case $(\bar{C}, \bar{W})$ represent the optimal trade-off between consumption growth and terminal wealth. The dotted curve illustrates how the indifference curve is twisted due to the certainty equivalence transformation of the parameters. However, we assume that the feasible set of (C,W) combinations is not influenced by the uncertainty, and the relevant indifference curve is thus $\tilde{U}$. A flatter indifference curve thus entails a change in the decision in the direction of a larger expected value of terminal wealth and a smaller expected value of consumption growth. This is indicated by the point (EC,EW) in figure 1, which represents the optimal trade-off between consumption growth and terminal wealth in terms of certainty equivalence. The parametric certainty equivalence procedure implies that the decision maker will tend to safeguard against uncertainty by taking a decision which implies a higher expected value of the variable which has the higher uncertainty, i.e. uncertainty as measured by the product of the standard deviation and the risk aversion coefficient.

The question we are addressing is what consequences should be inferred for current and future policy decisions from the uncertainty of national income. However, the conclusion suggested by the preceding analysis is partial in one important respect: Only the "substitution effect" of uncertainty was considered and not the "income effect". An implication of the static consideration is that the feasible set is given, i.e. expected income cannot be influenced by an appropriate reallocation of national wealth. Thus the larger uncertainty of terminal wealth induces an unambiguous substitution effect which makes the planner safeguard against uncertainty by taking a decision which implies a higher expected value of terminal wealth and viz. a lower expected value of consumption growth.

On the other hand, in a dynamic context, the appropriate question is to search for guidelines for reallocating national wealth in the view of uncertainty. An optimal composition of national wealth under uncertainty is obtained if risk-adjusted rates of return are equalized for all assets. That is, expected rates of return should be higher than the corresponding risk-free rate of return. In terms of expected values there is thus a potential gain from investments in uncertain assets, which is the source of the "income effect" of uncertainty. In the following section the dynamic optimization problem will be outlined, and in section 4 the complete income effect of uncertainty will be incorporated.

3 Static and dynamic optimization in a two-period context

Consider the following macroeconomic planning problem. At the outset the accumulated national wealth $\bar{W}_0$ is given, and the planning horizon comprises two periods ("now" and "future"). The planning problem is to decide on consumption in two periods, C_1 and C_2, under uncertainty of income in both periods. The utility of current consumption is weighed against the amount of wealth remaining at the end of period 2, W_2. The optimization criterion is the expected value of a sum-of-exponentials preference functions given by

$$(3.1) \quad F_0(C_1, C_2, W_2) = U(C_1) + U(C_2)/(1+\delta) + V(W_2)/(1+\delta)^2$$

where $U(C) = -B\exp(-\beta C)$ and $V(W) = -G\exp(-\gamma W)$.

δ is the rate of time preference.

The budget constraint for the three alternative uses of total wealth can be written

$$(3.2) \quad W_2 = (1+r)^2 (\bar{W}_0+R_1-C_1) + (1+r) (R_2-C_2)$$

where r is the certain rate of interest and R_i is income in period i. We shall assume that R_i is normally distributed with expectation $\bar{R}_i$ and variance σ_i^2.

We now apply the certainty equivalence property of the preference function (3.1).

$$(3.3) \quad \underset{C_1,C_2}{\text{Max}} \; E\left[F_0(C_1, C_2, W_2) \right] = \underset{C_1,C_2}{\text{Max}} \; F_0(C_1, C_2, \tilde{W}_2)$$

where $\tilde{W}_2 = EW_2 - 1/2\ \gamma\ \text{var}\ W_2$

$$= (1+r)^2(\bar{W}_o+\bar{R}_1-C_1) + (1+r)(\bar{R}_2-C_2) - 1/2\ \gamma\ (1+r)^2((1+r)^2$$

$$\sigma_1^2 + \sigma_2^2)$$

The stochastic optimization problem has been replaced by maximization of a non-stochastic function. The first order conditions for a maximum are

$$\text{(3.4a)} \quad U'(C_1) = ((1+r)/(1+\delta))^2\ V'(\tilde{W}_2)$$

$$\text{(3.4b)} \quad U'(C_2) = (1+r)/(1+\delta)\ V'(\tilde{W}_2)$$

from which the optimal values of C_1 and C_2 under <u>static optimization</u> can easily be derived. The solution is of the form

$$\text{(3.5)} \quad C_i = b\bar{W}_o + b(\bar{R}_1-1/2\ \gamma(1+r)^2\sigma_1^2) + b(\bar{R}_2/(1+r) - 1/2\ \gamma\sigma_2^2) + a_i$$

$$i = 1,2$$

i.e. consumption - in both periods - is a linear function of initial wealth and risk-adjusted income in both periods.

The consumption propensity b depends upon the risk aversion parameters, β and γ, and the rate of interest. The constant term is dependent upon the parameters of the preference function, the rate of interest and the rate of time preference. It can be seen directly from (3.4) that

$$\text{(3.6)} \quad C_2 - C_1 = \ln((1+r)/(1+\delta))/\beta \approx (r-\delta)/\beta$$

This growth formula for optimal consumption is in fact the same as under full certainty for the same preference function.

The static optimization problem as set out above pays no attention to the <u>strategic</u> problem faced by the long-term macroeconomic planner, which is the implication of taking sequential decisions under uncertainty of sequential events rather than taking all decisions at the outset of the planning period as assumed above. The strategic problem has two important aspects. One is that the scope for possible action at a future point of time may, and normally will, be narrowed down as a consequence of earlier actions and external influences. This may be due to irreversibility, capacity constraints, sluggishness etc. The other aspect is that future actions do not have to be taken until called for. This implies that future decisions can be based on more information than is available at the time of plan preparation, in particular, the realization of uncertain events in the period between the plan preparation and the decision point will be known. The problem is how to include this dynamic flexibility into an integrated plan.

The answer is to search for strategies, i.e. policy functions which are decision rules stating how policy decisions should be determined in each period on the basis of information available at the time. Perhaps the main purpose of long-term macroeconomic planning exercises should be the search for strategies. Unfortunately, the solution of this problem in the form of explicit policy functions are almost impossible to find except in very simplified cases.

We shall illustrate the difference between static optimization and the derivation of strategies by continuing our very simple example set out above toward an optimal dynamic solution.

In the dynamic solution the consumption decision in period 2 will be postponed until the beginning of that period, that is to say, it will be based on <u>known</u> wealth after period 1, W_1. The constraint for the remaining part of the planning period is

$$(3.7) \quad W_2 = (1+r)(W_1+R_2-C_2)$$

The second period decision is thus the solution to

$$(3.8) \quad \underset{C_2}{\text{Max}}\; E\left[F_1(C_2,W_2)\right] = \underset{C_2}{\text{Max}}\; F_1(C_2,\tilde{W}_2)$$

$$= \underset{C_2}{\text{Max}}\; \{U(C_2) + V(\tilde{W}_2)/(1+\delta)\}$$

where $\tilde{W}_2 = E_1 W_2 - 1/2\,\gamma\, \text{var}_1\, W_2$

$$= (1+r)(W_1+\bar{R}_2-C_2) - 1/2\,\gamma\,(1+r)^2\sigma_2^2$$

Note that $\tilde{W}_2$ here has a slightly different meaning than in (3.3). (E_1[ ] and var_1[ ] are expectation and variance at the end of period 1). The first order condition for maximum of (3.8) is

$$(3.9) \quad U'(C_2) = ((1+r)/(1+\delta))\, V'(\tilde{W}_2)$$

This equation determines C_2 as a linear function of W_1, $C_2^*(W_1)$.

$$(3.10) \quad C_2^* = b_2^* W_1 + b_2^*(\bar{R}_2 - 1/2\ \gamma(1+r)\sigma_2^2) + a_2^*$$

This is the strategy function for C_2. The optimal value of $\tilde{W}_2$ can likewise be written as a strategy function of W_1, $\tilde{W}_2^*(W_1)$, that follows directly from $C_2^*(W_1)$ and the expression for $\tilde{W}_2$.

The optimal first period decision can now in principle be found from (3.3) by replacing C_2 and $\tilde{W}_2$ by their strategy functions, i.e. the following problem:

$$(3.11) \quad \underset{C_1}{\text{Max}}\ E\left[F_0(C_1, C_2^*(W_1), \tilde{W}_2^*(W_1)) \right]$$

where $W_1 = (1+r)(\tilde{W}_0 + R_1 - C_1)$.

This way of proceeding leads, alas, to a problem we cannot solve explicitly. We must proceed by dynamic programming!

Let $J(W_1)$ be the optimal value of (3.8), i.e.:

$$(3.12) \quad J(W_1) = U(C_2^*(W_1)) + V(\tilde{W}_2^*(W_1))/(1+\delta)$$

From the first order condition (3.9) follows that $J(W_1)$ is proportional to $J'(W_1)$ which implies that $J(W_1)$ is an exponential function. The optimal C_1 can now be found by solving

$$(3.13) \quad \underset{C_1}{\text{Max}}\ \{U(C_1) + J(\tilde{W}_1)/(1+\delta)\}$$

where $\tilde{W}_1$ is risk-adjusted W_1 with the risk aversion coefficient βb_2^* of J().

The first order condition of (3.13) is

$$(3.14)\quad U'(C_1) = ((1+r)/(1+\delta))\, J'(\tilde{W}_1)$$

from which the explicit solution of C_1 can be found as

$$(3.15)\quad C_1^* = b\bar{W}_o + b(\bar{R}_1 - 1/2\ \gamma(1+r)^2\sigma_1^2 \frac{\beta}{\beta+\gamma(1+r)}) +$$

$$b(\bar{R}_2/(1+r) - 1/2\ \gamma\ \sigma_2^2) + a_1$$

In the dynamic solution the consumption decision for period 1 is related to wealth at the beginning of period 1, $\bar{W}_o$, and the risk-adjusted expected values for income in both periods.

How does the solution to the dynamic optimization compare with the static solution? We see that the only difference between (3.15) and the expression for C_1 in (3.5) is that the risk-adjustment for R_1 has been reduced and the optimal consumption is accordingly higher. The intuitive explanation for this is that under dynamic optimization - which means sequential decisions - optimal consumption comes out higher in the first period because less emphasis is put on the uncertainty of income in the first period. If this turns out to be different from expected income, it can to some extent be counteracted by the second period decision.

The answer given here is different from that which follows from the more widely known and applied assumption of a quadratic preference function. When combined with a linear model the answer in this case is given by the wellknown certainty equivalence result of Theil (1964): the first period decision on C_1 is the same in both cases and entails ~~no~~ risk adjustment.

What then about the second period consumption? In the strategy function for C_2 as given in (3.10) W_1 depends upon C_1 and realized income R_1. When C_1 is replaced by the right-hand side of (3.15) we arrive at the following expression for C_2

$$(3.16)\quad C_2 = b\bar{W}_o + b\bar{R}_1 + b(\bar{R}_2/(1+r) - 1/2\ \gamma\sigma_2^2) + \gamma(1+r)^2/(\beta+\gamma(1+r))$$

$$((R_1 - \bar{R}_1) + b\ 1/2\ \gamma\ (1+r)^2\sigma_1^2 \frac{\beta}{\beta+\gamma(1+r)}) + a_2$$

The only difference between this expression and that of C_2 in the static solution (3.5) is the effect of first period income. As C_2 in (3.16) includes the stochastic variable R_1, it may turn out to be smaller than the static solution, but the expected value of (3.16) is obviously higher than C_2 in (3.5), for two reasons: (1) there is no risk adjustment for income in period 1 and (2) by sequential decisions it is possible to consume from the "wealth reserve" that arises when the risk adjustment has been higher than the income loss. In short, by dynamic optimization the consumption decision in period 2 is based on more information: income in period 1 is not uncertain any longer. Thus strategies are worth searching for.

In the static optimization problem, (planned) consumption in period 2 increases according to the same growth formula as under full certainty, cf. (3.6). In the dynamic optimization problem, the increase in expected consumption in period 2 is somewhat higher. From the solution given by (3.15) and (3.16) we obtain (using (3.6))

$$(3.17) \quad EC_2 - C_1 = (1/\beta)\ \ln((1+r)/(1+\delta)) + (1/2)\ \beta\gamma^2(1+r)^4\ \sigma_1^2/(\beta + \gamma\ (1+r))^2$$

The gain from elaborating strategies for optimal consumption is here expressed by the risk-adjustment term of the growth formula. Using the definition of certainty equivalent consumption in period 2 we obtain

$$(3.18) \quad \tilde{C}_2 = EC_2 - (1/2)\ \beta\ \mathrm{var}\ C_2$$

$$= EC_2 - (1/2)\ \beta\ \gamma^2(1+r)^4\sigma_1^2/(\beta+\gamma(1+r))^2$$

and

$$(3.19) \quad \tilde{C}_2 - C_1 = (1/\beta)\ \ln((1+r)/(1+\delta)) \approx (r-\delta)/\beta\ .$$

In terms of certainty equivalent consumption, the dynamic optimization solution yields the same growth formula for consumption as under full certainty and in static optimization.

4 A multi-period framework with stochastic rates of return

We assume that the national wealth is distributed over a number of assets - physical and financial assets as well as natural resources. Assets are measured in terms of the purchasing power of consumption goods. The planning horizon is divided into periods of equal length. At the beginning of each period the returns on the various assets are added up and distributed between consumption and accumulation in the same assets. For the decisions to be taken at the beginning of each period we have the following budget equation

$$(4.1) \quad R_t = C_t + \sum_{i=0}^{n} I_{it}$$

where I_{it} is the investment in asset no. i and C_t is consumption in period t. Consumption is defined as the sum total of private and government consumption. All income is assumed to be capital income, accruing from investment undertaken one period earlier, hence

$$R_t = \sum_{i=0}^{n} r_{it} W_{it-1}$$

where W_{it-1} is the amount of asset no. i invested at the beginning of period t and r_{it} its rate of return. In asset terms the budget equation can be written

$$(4.2) \quad G_{t-1} = C_t + \sum_{i=0}^{n} W_{it}$$

where $G_{t-1} = \sum_{i=0}^{n} W_{it-1} + R_t = W_{t-1} + R_t$

Total wealth G_{t-1} at the beginning of period t hence consists of stocks of assets inherited from the past as well as capital income. The rates of return are stochastic variables. We assume that when decisions are to be made at the beginning of period t the outcome of the stochastic rates of return dated t is known with certainty whereas the uncertainty regarding future periods has to be taken into account. There is thus a minor difference here from the assumption made in the two-period model of the preceding section.

Oil reserves still in the ground can be considered as one type of assets although they are not usually counted as a part of national wealth. The value of the oil reserves can be measured as the product of the amount of reserves S_t and the price net of marginal extraction costs, $q_t = p_t - b_t$ where p_t is the current oil price and b_t is marginal extraction cost. We assume that marginal cost is constant with respect to the rate of extraction but is a hyperbolic function of the remaining reserves. The rate of return on the oil reserves is equal to the rate of growth of the net oil price.

Introducing oil as an additional asset in (4.2) and redefining total wealth G_t and total stock of assets W_t to include the oil reserves give

$$(4.3) \qquad G_{t-1} = C_t + W_t$$

$$\text{where } G_{t-1} = \sum_{i=0}^{n} W_{it-1} + R_t + q_t S_{t-1}$$

$$\text{and} \qquad W_t = \sum_{i=0}^{n} W_{it} + q_t S_t$$

The planning problem is now defined as the maximization of the sum of discounted expected utility from consumption over a planning horizon of length T, taking into consideration the discounted utility of terminal wealth. The utility of terminal wealth must be interpreted as derived from the consumption possibilities it represents beyond the planning horizon.

The objective function at the beginning of period t can thus be written as

$$(4.4) \qquad \sum_{\tau=t}^{T} U(C_\tau)(1+\delta)^{t-\tau} + V(G_T)(1+\delta)^{t-T-1} \qquad t = 1,2,\ldots T$$

For U() and V() we use the utility functions for current consumption and terminal wealth, respectively, introduced in section 3, and δ is the rate of time preference.

The decision problem at the beginning of each period is deciding on the reinvestment of total wealth and the level of consumption to be maintained in the period. The results of earlier decisions are represented in period t through the stock of assets inherited from the previous periods. We assume that total wealth can be frictionless reallocated between assets. The decisions to be taken in the following periods up to T have to be taken into account when deciding on consumption and investment at the beginning of period t. Decisions in all periods should reflect an appropriate trade-off between consumption and investment, as well as between consumption in the planning period and terminal wealth.

The optimization problem given by maximization of (4.4) under the budget constraint (4.3) and given initial values of oil stock and non-oil wealth can be solved by the method of stochastic dynamic programming which we applied in section 3. For a planning horizon starting at t=1 from given values of G_0 and S_0 the optimization problem is solved by beginning at the end of the planning horizon and solving the decision problem for each period recursively. At the beginning of period T the optimal W_{iT}, S_T and C_T are determined, given the initial condition G_{T-1} and S_{T-1}. Having found the optimal solution for the last period contingent on any initial condition G_{T-1} and S_{T-1}, we solve the two-period problem for the last two periods by choosing the optimal W_{iT-1}, S_{T-1} and C_{T-1}, contingent on the initial condition G_{T-2} and S_{T-2}, and so on. In the last stage the optimal W_{i1}, S_1 and C_1 are determined, given the initial values G_0 and S_0 available at the beginning of period 1. A crucial assumption for the optimality of this procedure is stochastic independence between rates of return, including the oil price, in different periods. Our approach follows Samuelson (1969) and Chow (1975).

In the notation of dynamic programming we denote the maximum expected value of (4.4), contingent on G_{t-1}, by $J_t(G_{t-1})$. The decision problem at the beginning of period t can now be more precisely stated as

$$(4.5) \quad J_t(G_{t-1}) = \text{Max } E\,\{U(C_t) + J_{t+1}(G_t)/(1+\delta)\}$$

where the maximization is with respect to the W_{it}'s and S_t and subject to (4.3). Before proceeding to the solution procedure, the stochastic assumptions must be specified.

The stochastic assumptions concerning future oil prices and rates of return are of considerable importance for the optimal solution. We shall assume that the rates of return of the various assets are multinormally distributed with expected values ϱ_1 and variances and covariances σ_{ij}, i,j=0,...,n. This implies that the standard deviation of income is proportional to the amount of capital and not constant as assumed in section 3. The oil price is assumed to be normally distributed with expected value π_t and variance τ^2. Covariances between the oil price and the rates of return on non-oil assets are given by τ_i, i=0,...,n. By the method of dynamic programming we start by solving the maximization problem given by (4.5) for t=T, i.e.

$$(4.6) \quad J_T(G_{T-1}) = \text{Max } E\,\{U(C_T) + V(G_T)/(1+\delta)\}$$

$$= \text{Max } \{U(C_T) + V(\tilde{G}_T)/(1+\delta)\}$$

where

$$\tilde{G}_T = EG_T - 1/2\ \gamma \text{ var } G_T$$

$$EG_T = \sum_{i=0}^{n} W_{iT}(1+\varrho_i) + (\pi_{T+1} - b_{T+1})S_T$$

$$\text{var}G_T = \sum_{i=0}^{n}\sum_{j=0}^{n} \sigma_{ij}W_{iT}W_{jT} + \tau^2 S_T^2 + 2\sum_{j=0}^{n} \tau_j W_{jT} S_T$$

Evaluating the terminal value of the oil reserves should take into account future oil price uncertainty beyond the planning horizon. The approach of measuring the terminal value by certainty equivalent net price at the beginning of period T does not capture this future uncertainty. However, the <u>marginal</u> value of the terminal oil reserves is equal to the certainty equivalent net oil price, provided that the terminal level of oil reserves is optimally weighed against consumption throughout the planning period and terminal stocks of non-oil assets.

The first order conditions for the solution of (4.6) are

$$(4.7a) \quad U'(C_T) = ((1+\tilde{r}_i)/(1+\delta))\ V'(\tilde{G}_T) \quad i=0,\ldots,n \text{ for non-oil assets}$$

(4.7b) $U'(C_T) = \tilde{q}_{T+1}/q_T \; V'(\tilde{G}_T)/(1+\delta)$ for the oil asset

where $\tilde{r}_i = \dfrac{\partial \tilde{G}_T}{\partial W_{iT}} - 1 = \rho_i - \gamma \sum_{j=0}^{n} \sigma_{ij} W_{jT} - \gamma \tau_i S_T \qquad i=0,\ldots,n$

and $\tilde{q}_{T+1} = \dfrac{\partial \tilde{G}_T}{\partial S_T} = \pi_{T+1} - b_{T+1} - \gamma \tau^2 S_T - \gamma \sum_{j=0}^{n} \tau_j W_{jT} - b'_{T+1}(S_T) S_T$

$\tilde{r}_i$ is the certainty equivalent rate of return on assets no. i, i.e. the marginal increase in certainty equivalent wealth by a marginal increase in asset no. i. $\tilde{q}_{T+1}$ is the certainty equivalent net oil price in period T+1. The difference between the certainty equivalent net oil price and the expected net oil price consists of the correction terms due to the uncertainty as well a term due to the dependence of marginal cost on the on the reserve level. With a hyperbolic marginal cost function, $b_t = m/S_{t-1}$, cost function terms in $\tilde{q}_{T+1}$ cancel out, and $\tilde{q}_{T+1}$ appears as

$$\tilde{q}_{T+1} = \pi_{T+1} - \gamma \tau^2 S_T - \gamma \sum_{j=0}^{n} \tau_j W_{jT}$$

To obtain an explicit solution for the optimal portfolio and consumption we make the crucial assumption that asset no. 0 is risk-free, yielding a certain rate of return r_0. Hence, $\tilde{r}_0 = r_0$ and from the first-order conditions we get

(4.8a) $U'(C_T) = ((1+r_0)/(1+\delta)) \; V'(\tilde{G}_T)$

(4.8b) $\tilde{r}_i = r_0 \qquad i=1,\ldots,n$

(4.8c) $\tilde{q}_{T+1}/q_T - 1 = r_0$

Optimal accumulation in the uncertain assets is determined by the condition that certainty equivalent rate of return should be equalized for all assets. Oil extraction is determined by a modified Hotelling rule: certainty equivalent net oil price should grow at a rate of return equal to the certain rate of return.

By proceeding as in section 3 we can solve the original problem recursively from starting at t=T and working backwards towards t=1. A more complete presentation of the solution procedure is given in Aslaksen and Bjerkholt (1984). The first order condition at t with respect to asset no. 0 is

$$(4.9)\quad U'(C_t) = ((1+r_0)/(1+\delta))\, J'_{t+1}(\tilde{G}_t) \qquad t=1,2,\dots,T$$

and we can show that

$$(4.10)\quad J'_t(G_{t-1}) = U'(C_t) \qquad t=1,2,\dots,T$$

Similar to what we found for the two-period model we find that the general form of the solution can be written as

$$(4.11)\quad C_t = b_{T-t}G_{t-1} + a_{T-t}$$

where $b_{T-t} = \gamma(1+r_0)^{T-t+1}/(\beta+\gamma \sum_{\tau=1}^{T-t+1}(1+r_0)^{\tau}) \qquad t=1,2,\dots,T$

and
$$a_{T-t} = -\beta\frac{1+r_0}{r_0}\frac{\alpha}{r_0} + \frac{b_{T-t}}{(1+r_0)^t}\left[(\ln(\beta B/(\gamma(1+r_0)G))+\chi)/\gamma(1+r_0) - \frac{t}{1+r_0}\left(\frac{\beta}{\gamma}-\frac{1+r_0}{r_0}\right)\alpha + \frac{1+r_0}{r_0}\frac{\alpha}{r_0}\right]$$

where $\alpha = (\ln\frac{1+r_0}{1+\delta} - \chi)/\beta \approx (r_0-\delta^*)/\beta$

$\delta^* = \delta+\chi$

and
$$\chi = 1/2\sum_{i=1}^{n}\sum_{j=1}^{n}(\rho_i-r_0)(\rho_j-r_0)\dot{\sigma}_{ij} + \left(\frac{\bar{q}_{t+1}}{q_t} - 1 - r_0\right)\sum_{j=1}^{n}(\rho_j-r_0)\dot{\tau}_j + 1/2\left(\frac{\bar{q}_{t+1}}{q_t} - 1-r_0\right)^2\dot{\tau}^2$$

$\dot{\sigma}_{ij}$, $\dot{\tau}_j$ and $\dot{\tau}^2$ are the elements of the inverse of the variance-covariance matrix of σ_{ij}, τ_j and τ^2, and $\bar{q}_{t+1}$ is the expected net price (equal to $\pi_{t+1}-b_{t+1}$). As an implication of the certainty equivalence procedure, the stochastic parameters appear only in the risk-adjusted time preference rate δ^*.

The marginal propensity to consume out of current wealth, b_{T-t}, is dependent upon the risk aversion coefficients γ and β as well as the interest rate r_0. It is in fact only the ratio between γ and β that matters. In the limiting case where $T-t \to \infty$, b_{T-t} approaches a constant given by

$$b = r_0/(1+r_0)$$

In this case the marginal propensity to consume is independent of γ as well as β. However, γ and β appear in the constant term of the consumption function.

When the optimization problem has been solved step by step, optimal consumption is implemented by recording actual development and inserting, period by period, the outcome of the stochastic rates of return, as expressed by G_{t-1}, in the consumption function (4.11). The optimal solution can thus be interpreted as a <u>strategy</u>; decision rules for optimal consumption are calculated initially, whereas actual consumption decisions are postponed until current wealth is known with certainty.

As in the two-period model, it is easily demonstrated that certainty equivalent consumption increases according to the same growth formula as under full certainty. From (4.9) and (4.10) follows

$$U'(C_t) = ((1+r_0)/(1+\delta))\, U'(\tilde{C}_{t+1})$$

hence

$$\tilde{C}_{t+1} - C_t = \ln((1+r_0)/(1+\delta))/\beta \approx (r_0-\delta)/\beta \tag{4.12}$$

Given the optimal consumption, the accumulation in the uncertain assets is determined as a one-period portfolio problem

$$W_{it} = \frac{1}{\beta b_{T-t}} \{ \sum_{j=1}^{n} (\varrho_j-r_0)\dot{\sigma}_{ij} + \dot{\tau}_i(\pi_{t+1} - (1+r_0)q_t)\} \tag{4.13}$$

$$(4.14)\quad S_t = \frac{1}{\beta b_{T-t}} \{ \sum_{j=1}^{n} (\varrho_j - r_0)\dot{\tau}_j + \bar{\tau}^2 (\pi_{t+1} - (1+r_0)q_t) \}$$

Hence, optimal oil extraction in period t is given by

$$(4.15)\quad X_t = S_{t-1} - S_t$$

where S_t is determined by (4.14) and S_{t-1} is given from the previous period.

Due to the strong assumptions regarding the utility function and the stochastic parameters as well as the production structure and the cost function for oil extraction we have thus obtained explicit solutions with intuitive interpretations.

5 Preference functions derived from macroeconomic projections

An empirical application of the stochastic optimization framework requires an assessment of the risk aversion coefficients. In this context we have approached this problem by the method of deriving the underlying preferences from macroeconomic projections currently elaborated and presented.

In Norwegian economic planning long-term macroeconomic projections are usually elaborated in connection with the quadrennial government White Papers presenting a four-year plan and a less detailed and less committing projection for the ensuing 20-30 years. The purpose of such projections is threefold. The shall serve

- as the basis for government policies over a wide range of issues,
- as guidelines for the development of the national economy that can be linked to sectoral, regional and other less comprehensive analyses, and
- as a general orientation about the economic prospects for the public at large.

All these projections have been elaborated by means of successive versions of the MSG model, originally constructed by Leif Johansen in 1960. The MSG model is a large general equilibrium model which combines an overall macroeconomic framework a with considerable amount of details. The model has been extensively presented elsewhere in this book and will not be further discussed here.

One of the more difficult tasks in the elaboration of long-term projections is to account properly for the many aspects of inherent uncertainty in the preparation and presentation of future development paths. With a time span of twenty or more years ahead there are large amounts of uncertainty with regard to many of the exogeneous assumptions on which the analysis is based. Greater efforts of gathering information could probably reduce this uncertainty to some degree, but much would still remain. For a small open economy much of the uncertainty stems from abroad, such as the growth in world trade and the future crude oil price.

The traditional ways of dealing with such uncertainty are either to present alternative broad scenarios or to use sensitivity calculations varying the assumptions about exogenous influences. Such methods can give interesting illustrations of the uncertainty. But in a planning context the important question is what conclusions can be drawn for current and future planning decisions from this uncertainty. The uncertainty as it propagates from the exogenous influences must be evaluated in view of what can be governed or influenced by means of economic policy.

The development of the oil sector in the Norwegian economy has entailed a considerable macroeconomic exposure to risk, and the need for an explicit consideration of uncertainty issues is thus more strongly felt today than earlier. Most of the attention given to uncertainty in connection with the increased reliance upon petroleum extraction in the Norwegian economic and political debate has been related to short- and medium-term consequences of a volatile oil price. Less attention has been given to uncertainty in the longer term perspective. However, two recent reports from government appointed committees have i.a. dealt with these perspectives (NOU 1983 : 27, NOU 1983 : 37).

Our work is related to that of these two committees and may be regarded as suggestions of how the analyses could be brought further. We are well aware that answers given are very tentative to say the least, both theoretically and empirically. Our own attitude to them can be well expressed by a quote from Leif Johansen (who in fact initiated our work on this topic) who wrote in the introduction to his book on the MSG model: ".... if I were required to make decisions and take actions in connection with relationships covered by this study I would (in the absence of more reliable results, and without doing more work) rely to a great extent on the data and the results presented in the following chapters." (Johansen, 1960).

The intended application of the stochastic optimization framework outlined above is mainly as a means for evaluating and corroborating long-term projections from the MSG model. Although stochastic elements are not included in the MSG model, the model is a valuable means for illustrating the wide range of possible long-term projections under alternative oil price assumptions. Model calculations are performed with alternative oil price scenarios and exogenously stipulated oil and gas production profiles. The consequences of alternative oil revenue scenarios are traced out by model calculations. These long-term projections illustrate the considerable impact on sectoral development and accumulated foreign reserves under alternative oil price assumptions. A consistent evaluation of these long-term equilibrium growth paths under uncertainty requires a stochastic optimization framework.

Our emphasis is not on the treatment of uncertainty in macroeconomic projections in general, but rather on the implications of uncertainty for the selection of "optimal" or "good" paths.

Our analysis is based on projections in a report called the "Perspective Analysis" (NOU 1983:37), published in 1983 by an appointed committee of experts relying to a great extent on the model tools and data sources used by the government for its projections. The committee stated views on the methodology of using macroeconomic models for long-term projections as well as presenting its own projections in the form of a reference path and alternative scenarios reflecting both uncertainty issues, policy alternatives and policy performance. The methodological part included remarks on how to cope better with uncertainty in macroeconomic projections, but refrained from introducing new procedures in the preparation and presentation of projections compared to earlier government projections.

The projections of the Perspective Analysis were elaborated without the political commitments that are given to the projections presented in e.g. the quadrennial medium-term programme. However, for our purpose it may not be totally misleading to interpret them as reflecting current political preferences. The projections of the Perspective Analysis do not easily lend themselves to the assessment of preferences. Little is said about the evaluation of the alternative projections, and no precise guidelines are given for the trade-off between consumption and wealth accumulation.

Although no explicit welfare function or preference indicator has been applied in the elaboration of the projections, the various statements given in the report can be interpreted as expressions of a set of underlying preferences. The discussion of the policy choices between domestic use of oil revenues versus increased current account surplus has been our starting point for deriving the preferences.

The present analysis is based on the reference path and the four alternative projections which are summarized in table 1.

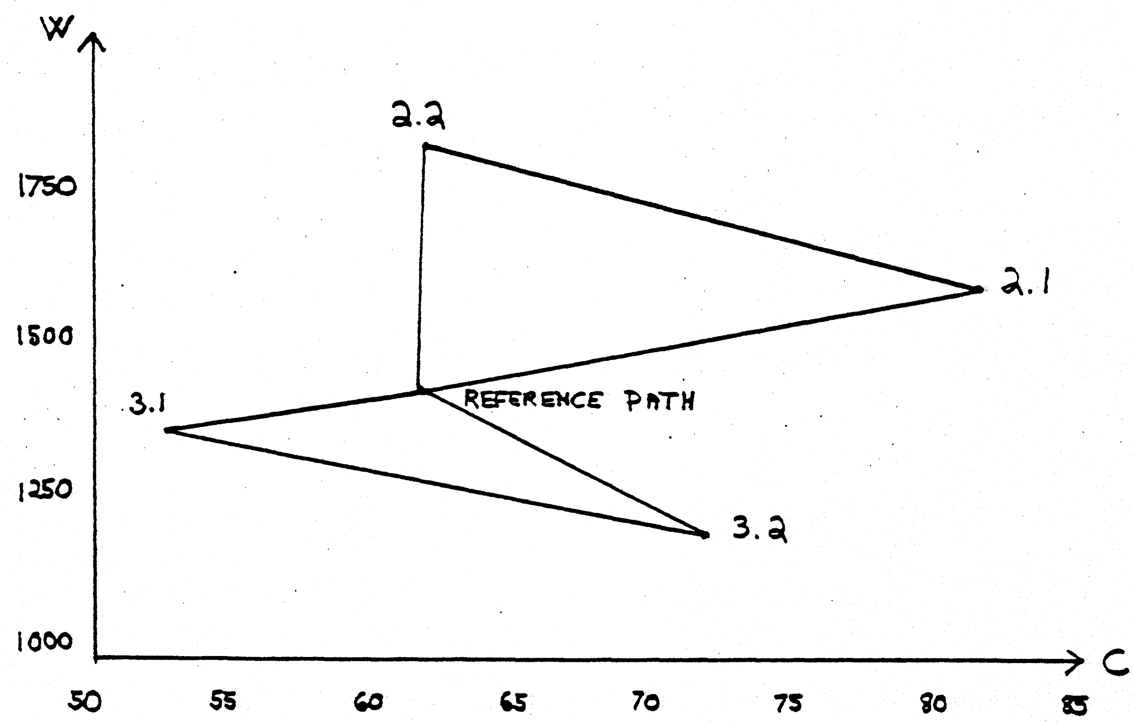
Table 1. Selected results from the Perspective Analysis.

Scenario	Total consumption (private and government) in 2000 as increase over 1980. Percent	Net foreign reserves in 2000 plus value of proven oil reserves in 2000. Billion kroner
1. Reference path	62.2	1419.6
2. Higher petroleum income		
2.1 Increased domestic use	82.8	1575.4
2.2 Increased capital exports	62.2	1811.8
3. Sluggish world economy		
3.1 Tight policy	50.5	1379.4
3.2 Lax policy	70.6	1204.9

Note: The figures are derived from NOU 1983:37, and unpublished material from the Ministry of Finance. The reference path is based on full employment and an increase in the production of oil and gas reaching 80 mill. toe in year 2000. The crude oil price is in the reference path assumed to grow with 2 percent p.a. in real terms. Non-oil export grows with less than 2 percent p.a. In the two higher petroleum income scenarios the production of oil and gas is assumed to reach 90 mill. toe in year 2000, while the crude oil price grows with 3 percent p.a. In 2.1. the increased income is used to promote growth in domestic demand. Employment and the rate of technical progress increase, while in 2.2 the increased income is accumulated as foreign assets. The sluggish world economy scenarios depict developments where non-oil exports grow even less than in the reference path, only 1 percent p.a. In 3.1 the balance of payments is maintained by means of tight demand management. Employment falls off compared with the reference path. In 3.2 on the other hand priority is given to employment. Private and Government Consumption are increased with adverse consequences for the balance of payments. This table reveals, in fact, little about the differences between the alternatives. The Perspective Analysis also presented 3-4 other alternative scenarios.

These alternative projections of the Norwegian economy toward year 2000 results in different states of the economy by the end of the planning period. In the highly simplified representation of these alternatives in our further discussion we ignore most structural and other aspects of the differences between these alternatives and focus on only two variables: consumption level (or rather increase over 1980) and wealth position. Figure 2 plots all five projections with regard to these two characteristics.

Figure 2. Selected projections from the Perspective Analysis. Percentage increase in total consumption in 2000 over 1980 (C) and accumulated wealth in 2000 (W).



C = Total consumption (private and government) in 2000 as percentage increase over 1980.

W = Net foreign reserves in 2000 plus value of proven oil reserves in 2000 (see table 1).

These five projections illustrate a wide range of possibilities for the choice between consumption and accumulation of foreign assets. The two triangles in figure 2 indicate the feasible sets under the assumptions of either higher petroleum income (2.1 and 2.2) or sluggish world economy (3.1 and 3.2). Little is said about the choice between increased domestic use and increased capital exports in the case of higher petroleum income, and the choice between tight and lax policy in the case of a sluggish world economy.

Based on the information provided in the report of the Perspective Analysis we have however established the following crucial assumptions.

Consider the following stochastic experiment with two possible outcomes: Either the outcome of higher petroleum income is realized, where the feasible set is represented by the line segment between 2.1 and 2.2, or the outcome of a sluggish world economy is realized, where the feasible set is represented by the line segment between 3.1 and 3.2. These two outcomes are assumed to have an equal probability. The alternatives 2.1 or 2.2 and 3.1 or 3.2 thus represent extreme policies under each income scenario, and to reveal the optimal policy we state the following assumptions:

a) Sluggish world economy: Given a feasible set of all points between 3.1 and 3.2 the best choice is to pursue a policy aiming at a result midway between the two extreme policies.

b) Higher petroleum income: Given a feasible set of all points between 2.1 and 2.2 the best choice is to pursue a policy aiming at a result slightly closer to 2.1 than the midpoint.

c) Reference path: The reference path is considered as the certainty equivalent of the stochastic experiment described above. Given the two optimal policies described in a)-b) the expected utility of these two outcomes is equal to the utility of the reference path.

These assumptions are formulated in view of a preference function given by

(5.1) $U(C,W) = -B\exp(-bC) - G\exp(-gW)$

C = Total consumption (private and government) in 2000 as percentage increase over 1980.

W = Net foreign reserves in 2000 plus value of proven oil reserves in 2000 (see Table 1).

To simplify the estimation of the risk aversion coefficients, the preference function (5.1) has been formulated as a static analogy to the multi-period preference function (4.4) of the dynamic optimization problem. The implications of a preference function like (5.1) are more extensively discussed in section 2. In (5.1) preferences are attached to the percentage increase in consumption over the planning horizon, rather than the sum of discounted utility of consumption in each period. However, this reformulation does not alter the main conclusions for the trade-off between consumption and terminal wealth under uncertainty. The numerical estimate for the risk aversion coefficient b will differ from the risk aversion coefficient β of the multi-period preference function, and the appropriate estimate for β will finally be derived.

The wealth concept W defined as net foreign reserves plus the value of the oil reserves is highly tentative, to say the least. It does not properly reflect the concept of national wealth as defined in the optimization model. According to the preference function (4.4), consumption should be weighed against total wealth at the end of the planning period, i.e. production capital, financial assets and natural resources. The role of terminal wealth in the preference function is to represent the production and income potential for future consumption beyond the planning horizon. The discussion of the Perspective Analysis is however more explicitly related to the trade-off between consumption growth and net foreign reserves at the end of the planning period. The point of foreign reserves in this connection seems to be as a safeguard against the risk inherent in the oil reserves. In order to accommodate the views expressed in the report as a guideline for our estimation of the risk aversion coefficients, the value of petroleum reserves and net foreign reserves are included in our wealth concept here while other assets are disregarded. This is perhaps a dubious interpretation and inclusion of real capital would have given different estimation results.

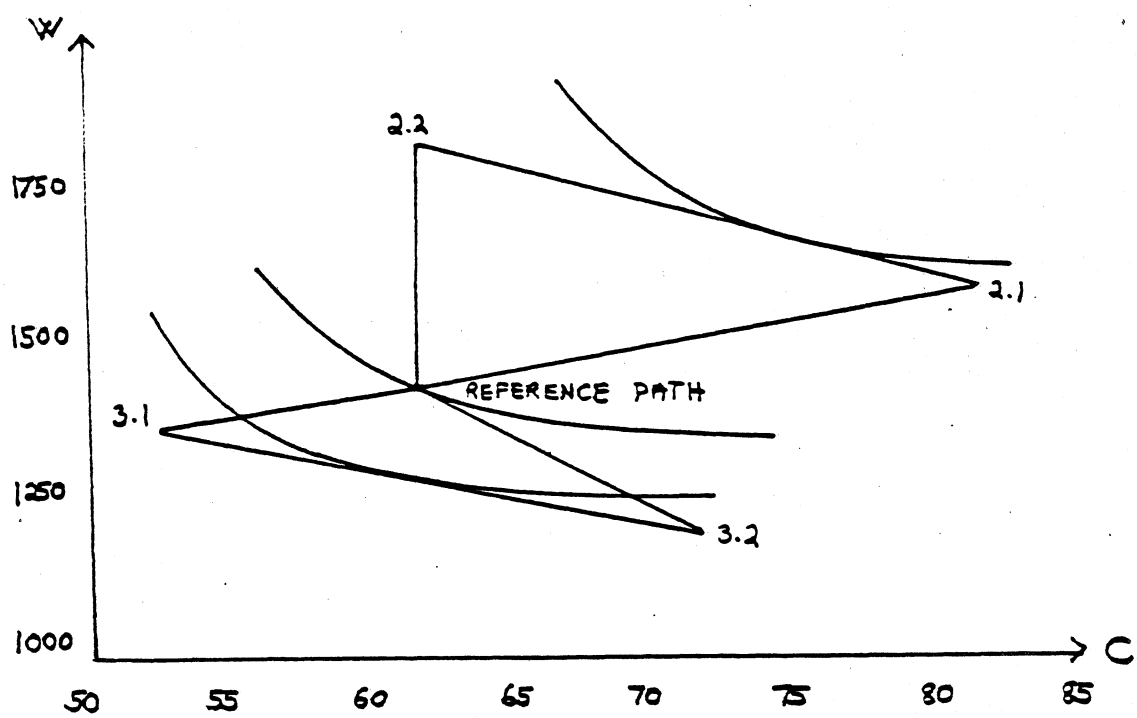
The assumptions a)-c) give three relationships to determine the parameters b, g and G/B. The level of utility is arbitrarily chosen by setting B=1. Furthermore, the parameter values are adjusted to yield G=B=1. The following parameter values are thus obtained:

$b = 0.1426$

$g = 0.00589$

Figure 3 shows the estimated preference function as represented by three indifference curves implicitly referred to in assumptions a)-c).

Figure 3. Indifference curves with b=0.1426 and g=0.00589.



In order to apply the multi-period framework of section 4, we have to establish the correspondence between the preference function (4.4) of the dynamic model and the static analogy given by (5.1). In the dynamic model which is to be applied in section 6, preferences are formulated in terms of the sum of discounted utility from consumption over the planning period, whereas in the static preference function (5.1), the relevant concept is percentage increase in consumption over the planning horizon. In order to find the appropriate risk aversion coefficient in a dynamic context, we make the assumption that the sum of discounted utility from consumption over the planning period is equal to the utility of the percentage increase of consumption. The annual growth rate of consumption in the reference path is 2.4 percent. We assume that the time preference rate is 1 percent. Given the estimate of b=0.1426 an estimate of β=0.0352 is thus obtained for the risk aversion coefficient of the dynamic model. The estimate of the risk aversion coefficient g=0.00589 is calibrated in order to include the production capital. An estimate of γ=0.0027 is thus obtained.

6 A strategy for optimal consumption under uncertainty: empirical applications

In the preceding section we looked at actual long-term projections of the Norwegian economy and tried to estimate the risk aversion coefficients that seemed to be implicit in the considerations of the committee responsible for the projections. An application of certainty equivalence procedures in establishing long-term projections would, of course, imply an integration of the ideas set out in this article at a much earlier stage in the elaboration of projections. We shall nevertheless attempt a tentative application within the framework and scenarios of the Perspective Analysis.

The point of departure is the formal model framework of section 4. First we shall define the composition of total wealth as required in the model. We shall distinguish between four assets apart from oil:

W_S = real capital in the sheltered sector (i.e. non-tradable goods production, protected sectors, and government)

W_E = real capital in the export sector

W_H = real capital in the import-competing sector

W_U = foreign assets

Table 2. Average rates of return.
Estimation period 1962-1981

ϱ_i	Percent
Sheltered sector (excluding government)	7.53
Import-competing sector	10.00
Export sector	5.45

Foreign assets are assumed to yield a risk-free rate of return of 3 percent. This is our r_0. The increase in expected net oil price, is assumed to be 2 percent p.a. through the whole period.

The variance-covariance matrix for the estimated rates of return in the period 1962-81 is given in table 3 and the inverse variance-covariance matrix is given in table 4.

Table 3. Variance-covariance matrix. Estimation period 1962-1981

$\sigma_{ij}, \tau_i, \tau^2$	Sheltered sector	Import-competing sector	Export sector	Real oil price
Sheltered sector	1.65685	-0.088861	-1.84331	-1.28275
Import-competing sector		2.30443	1.93291	-0.897742
Export sector ..			13.8807	0.621457
Real oil price .				16.789

Table 4. Inverse variance-covariance matrix.

$\dot{\sigma}_{ij}, \dot{\tau}_i, \dot{\tau}^2$	Sheltered sector	Import-competing sector	Export sector	Real oil price
Sheltered sector	0.756811	-0.037254	0.102251	0.052005
Import-competing sector		0.507531	-0.076836	0.027137
Export sector ..			0.096458	0.000218
Real oil price .				0.064979

The choice of breakdown of non-oil national wealth is - as the other specifications of the model - rather tentative. A priori we would expect capital in the non-tradeable sector to be a more certain asset (i.e. lower rate of return, but also lower variance) than investment in the tradeable sectors, while foreign reserves are assumed to be a risk-free asset.

For a small oil exporting country like Norway the oil price is exogenous, independent of domestic reserves and rate of extraction. It may be less obvious that the stochastic rates of return on assets other than oil are independent of time and of the stocks of the respective assets, as assumed in section 4 above. In the following we assume that real capital by sector has constant expected rates of return as set out above. This exceedingly simplified picture of a national economy can only be defended on the ground that it serves a higher purpose!

For the risk aversion coefficients we use those derived in section 5. For the rate of time preference we assume $\delta = 0.01$.

From these estimates and assumptions we can calculate a reference scenario based on the stochastic optimization model. There are many reasons why this reference scenario will not coincide with the reference scenario of the Perspective Analysis elaborated by means of the MSG model.

The stochastic optimization model has hardly any macroeconomic infrastructure. The labour market, production structure and foreign trade are not explicitly dealt with. The asset composition can be changed in a frictionless way, we thus pay no attention to the transition problem of changing the asset composition from what is historically given. An additional problem is the more specific assumptions of the constancy of the parameters of the model estimated above. We have chosen, however, to interpret the scenario based on these assumptions as the appropriate scenario for the further analyses. In the reference scenario consumption increases smoothly and reaches a level in 2000 which is 74.4 percent higher than in 1980. Investments in uncertain assets are declining throughout the planning horizon, whereas foreign debt is gradually reduced. Total wealth is increasing in early years and is decreasing thereafter. The development of total wealth is crucially dependent on the relationship between the risk aversion parameters and the risk-free rate of return. The parameters of our reference scenario give the condition

$$(6.1) \quad \beta/\gamma < (1+r_0)/r_0$$

which entails the concave path of total wealth as shown in figure 5. If the inequality sign of (6.1) is reversed, total wealth increases along a convex path.

Optimal oil reserves in 1980 turn out somewhat lower than the actual level of oil reserves in 1980. An initial jump in the oil production profile to 161 million toe is thus necessary in order to reach the optimal path which starts at 106 million toe in 1981 and increases gradually to 123 million toe in 2000.

The numerical results obtained by the stochastic optimization model are highly dependent on the choice of parameter values. Table 5 illustrates the effects on consumption, foreign reserves and terminal wealth of partial shifts in the parameters of the reference scenario.

A partial increase in β implies reduced consumption in all periods at least when the planning horizon is 20 years. In the optimal strategy for consumption, the marginal propensity to consume out of current wealth decreases with β (not shown in the table), in the two-period as well as in the multi-period context, and this wealth effect dominates any positive effect through the certainty equivalence correction terms. Lower consumption throughout the planning horizon entails higher total terminal wealth. Investments in uncertain assets decrease with β, because the certainty equivalent rates of return are reduced and it is less profitable to invest in uncertain assets. However, in the last period of the planning horizon, the increase in β has no effect on the level of investment in the uncertain assets - in the last period it is γ which is the risk aversion parameter for deriving the certainty equivalent rates of return and hence optimal investments in the uncertain assets. The reduced accumulation in uncertain assets implies higher foreign reserves.

Table 5. Effects of positive shifts in the parameters

	C_t 1980	C_t 2000	G_{2000}	W_{Ut} 1980	W_{Ut} 2000
Parameter:					
β	-	-	+	+	+
γ	+	+	-	+	+
τ	≈0	+	+	+	+
π_{t+1}/π_t	≈0	+	-	-	-
ϱ_S	≈0	+	+	-	+
ϱ_{HK}	≈0	+	+	-	+
ϱ_{UK}	≈0	+	+	-	-
r_0	+	-	-	+	-
δ	+	-	-	-	-

An increase in γ has the opposite effect on consumption in all periods and terminal wealth as compared to an increase in β. This has a quite intuitive appeal in view of the dynamic programming solution of the model whereby the optimal trade-off between consumption and remaining wealth is established in each period recursively. However, increases in γ and β both tend to reduce investment in uncertain assets through the certainty equivalence correction terms. In order to maintain the higher consumption entailed by an increased γ, investment in foreign assets must increase toward the end of the planning horizon.

An increase in the variance of the oil price implies that investments in all uncertain assets are reduced, because the increased uncertainty makes it less profitable to invest in uncertain assets and more wealth is converted into the risk-free asset. Moreover, consumption is higher throughout the planning period.

An increase in any of the expected rates of return, including the expected growth rate of net oil price, provides a positive income effect which increases consumption in all periods as well as terminal wealth. A higher expected rate of return naturally implies increased investment in the asset in question, and the effect on the other uncertain assets depends on the signs of the covariances. The effect on investing in foreign assets is initially negative, for later periods the sign varies.

The role of the risk-free rate of return r_0 is somewhat complicated. The potential gain from choosing uncertain alternatives in the allocation of national wealth is less the higher r_0 is compared to the expected rates of return. Investments in all uncertain assets are thus reduced when r_0 increases. This income loss gives less scope for investing in foreign assets, and it is therefore not so that an increase in the risk-free rate of return implies higher investment in the risk-free asset. Naturally, terminal wealth is reduced. Consumption is somewhat higher in the first periods and is later reduced due to the loss of the potential gain from uncertainty.

An increase in the rate of time preference δ implies that the consumption path is shifted toward the beginning of the planning period. This early increase in consumption gives less scope for saving and terminal wealth is reduced. Investments in uncertain assets are not affected by δ since the allocation of total wealth between uncertain alternatives is determined as a static portfolio problem once the optimal trade-off between consumption and future wealth has been established in each period.

An important question in macroeconomic planning under uncertainty is to elucidate the implications for economic policy with given stochastic assumptions and an explicitly stated attitude towards risk. The scenarios of the Perspective Analysis which we are referring to illustrate the implications of tight and lax economic policy under alternative scenarios for uncertain future income. These policy alternatives are established by variations in the exogenous variables and government instruments in the MSG-model. It is not obvious how a corresponding simulation of policy alternatives can be performed in the stochastic optimization model. However, different assumptions about the risk aversion parameters entail a different propensity to pursue a tight or a lax economic policy. Consider a situation where the government is more concerned about future consumption possibilities and wants to pursue a policy for increasing the national wealth at the end of the planning period. In a fully elaborated macroeconomic context this is aimed at by steering the exogenous variables so as to decrease current consumption and promote accumulation in foreign (and domestic) assets. In our model it is natural to express such a concern in terms of risk aversion: a policy which aims at reducing current consumption and increasing terminal wealth corresponds to a shift in the risk aversion parameters toward higher β and lower γ.

In order to realize the effect of risk aversion, it is elucidating to express a specified change in economic policy by the corresponding variations in the risk aversion parameters. In this empirical application we have intended to interpret the policy alternatives discussed by the Perspective Analysis in terms of risk aversion.

The alternatives 2.2 and 3.1 represent the tight policy alternative under each income scenario, whereas 2.1 and 3.2 represent the lax policy alternatives. By variations in β and γ we have established the scenarios of our model which correspond to these four policy alternatives. The results are presented in table 6 and in figure 4 and 5.

As the reference scenario of the stochastic optimization model deviates from the reference scenario of the MSG-model, it has not been our intention to simulate the four alternatives (2.1, 2.2, 3.1, 3.2) exactly by appropriate adjustments of the risk aversion parameters. However, we have applied the same criteria for establishing our alternatives as those of the Perspective Analysis:

- Tight policy in the high expected income scenario (=2.2) and lax policy in the low expected income scenario (=3.2) should aim at the same increase in consumption as the reference scenario

- Tight policy in the low expected income scenario (=3.1) and lax policy in the high expected income scenario (=2.1) should aim at the same terminal wealth position as the reference scenario.

Table 6. Selected results from the stochastic optimization model.

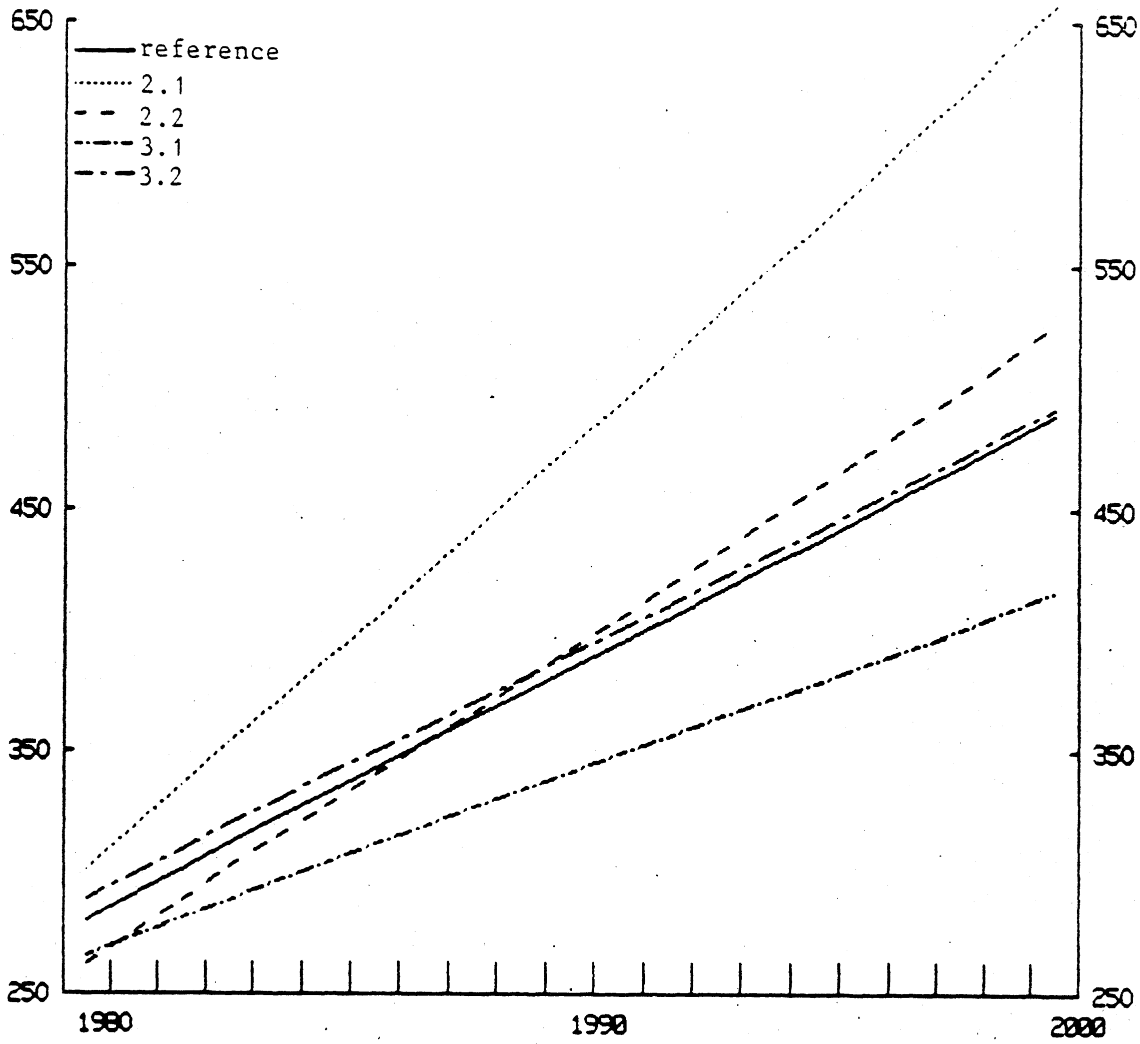
Scenario	Total consumption in 2000 as increase over 1980. Percent	Total wealth 2000 Billion kr	Net foreign reserves 2000 Billion kr
1. Reference path	74.4	5568	-1219
2. Higher petroleum income			
2.1 Increased domestic use	118.2	5561	-973
2.2 Increased capital exports ...	100.0	7846	-878
3. Sluggish world economy			
3.1 Tight policy ..	56.3	5619	-1218
3.2 Lax policy	70.2	4904	-1911

These criteria have been our guidelines for the choice of appropriate variations of the risk aversion coefficients. The numerical results can be summarized as follows:

An increase in β of 15 percent corresponds to a tight policy whereby consumption is reduced according to this criterion. Since the terminal wealth position differs between the two income scenarios, it was furthermore necessary to reduce γ by 10 percent to simulate 2.2 and to reduce γ by 5 percent to simulate 3.1. Terminal wealth varies inversely with γ. Since 2.2 is the alternative of higher expected income than 3.1, it was necessary to reduce γ relatively more in the simulation of 2.2 to account for the large terminal wealth of this alternative.

In the simulation of the lax policy alternatives 2.1 and 3.2 β was decreased accordingly, i.e. by 15 percent. Since the terminal wealth position of 2.1 should reflect the higher expected income as compared to 3.2, γ is increased by 20 percent in 2.1 and γ is decreased by 5 percent in 3.2.

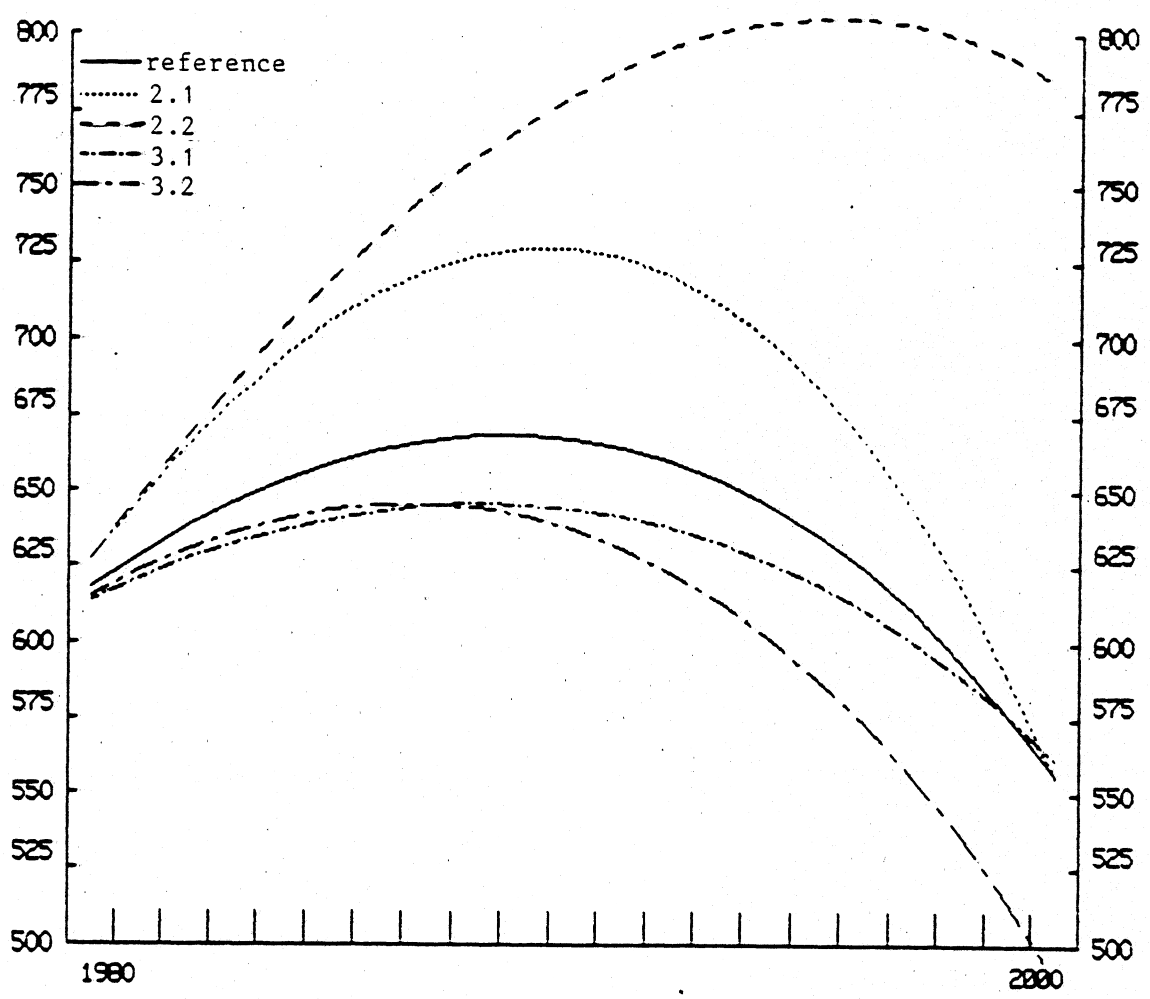
Figure 4. Consumption path 1980-2000. Billion kroner



The optimal consumption paths are illustrated in figure 4. The simulations of the lax policy alternatives, 2.1 and 3.2, give consumption paths which are higher than that of the reference scenario throughout the planning horizon. The consumption path of the tight policy alternative 3.1 is accordingly lower. However, the consumption path of 2.2, which simulates tight policy in the high income scenario, intersects the consumption path of the reference scenario. The appropriate increase in risk aversion entails lower consumption in early years, and this effect is gradually reversed due to the wealth effect of the high income assumption.

The variations in the risk aversion coefficients β and γ in the simulation alternatives are all within the range given by (6.1), thus total wealth increases in early years and declines thereafter, see figure 5. In accordance with our criteria for establishing the simulation alternatives, terminal wealth in the scenarios of 2.1 and 3.1 turns out as in the reference path, while terminal wealth of 3.2 is lower and terminal wealth of 2.2 is higher. The initial decline in consumption in the high income scenario 2.2 gives a considerable growth in total wealth, and the sloping down of total wealth occurs at a later stage than in the other alternatives. Thus consumption in 2000 comes out somewhat higher than in the reference scenario.

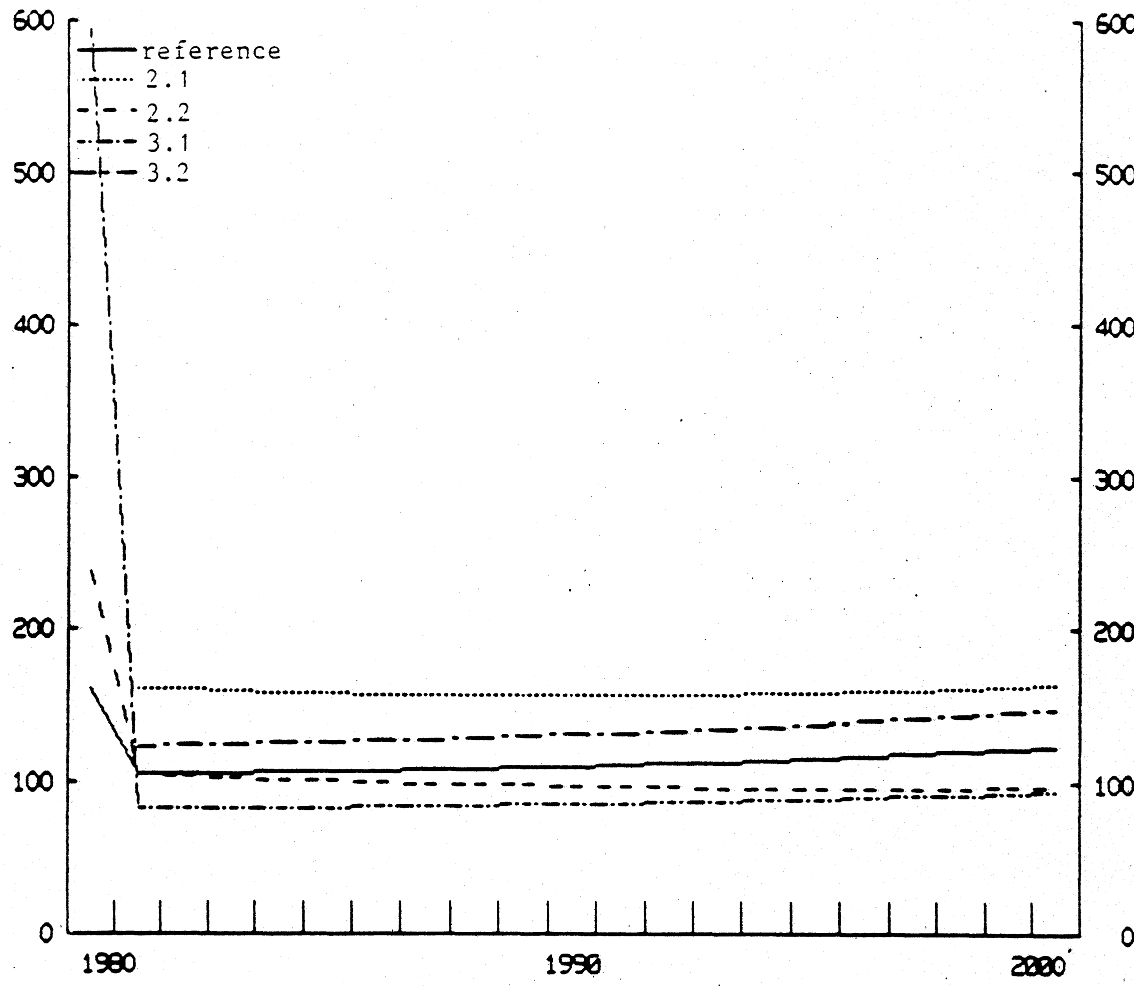
Figure 5. Total wealth 1980-2000. 10 billion kroner.



Investments in uncertain assets decline monotonically in all simulation alternatives, as our assumptions about the risk aversion parameters entail that a conversion from uncertain assets into the risk-free alternative will take place. Thus foreign debt is reduced throughout the planning period in all alternatives. However, none of the alternatives imply positive net foreign reserves within a horizon of 20 years. The explanation is that initial optimal accumulation in the uncertain assets is substantially higher in all alternatives than the corresponding national account figures for 1979, which are used to determine initial wealth of the model. In order to realize the optimal solutions for consumption and investments in uncertain assets a substantial foreign debt has to be incurred initially. Net foreign wealth position is determined residually in the model, and the initial levels of foreign debt entail that positive net foreign reserves cannot be obtained in a time span of 20 years. This conclusion is crucially dependent on the constancy of the other parameters but the risk aversion coefficients. If e.g. the standard deviation of the oil price is increased by 25 percent, a decline in the accumulation of uncertain assets whereby foreign debt starts at a more modest level than in the depicted alternatives. With this assumption the reference scenario will in fact come out with positive net foreign reserves in 2000.

Within each alternative oil production is fairly constant once the optimal path has been reached, see figure 6. In the alternatives of lax policy 2.1 and 3.2, the initial optimal value of the oil reserves is substantially higher than the initial estimate for the value of the oil reserves because of the reduction in the risk aversion coefficient β. As a consequence, the model gives negative oil extraction in 1980 (not shown in the figure). Similarly, a substantial peak occurs in initial oil production in the tight policy alternatives 2.2 and 3.1 because the initial optimal value of the oil reserves is reduced due to the higher degree of risk aversion. However, from 1981 the oil production paths show that optimal oil production is higher in the lax policy alternatives and lower in the tight policy alternatives as compared to the reference scenario.

Figure 6. Oil depletion. 1980-2000. Mtoe.



Notes: For scenario 2.1 and 3.2 the first-year depletion is negative, about -320 mtoe.

Although solutions with initial peaks in oil production or initial negative oil production are not exceptional in an optimization context, they can naturally not be implemented. In terms of policy guidelines, we can, however, interpret the initial negative oil production as an indication that oil production has been too high in the preceeding periods and the remaining level of oil reserves is too small in 1980 as compared to the optimal oil reserves. If oil production is temporarily postponed for some years, the value of the oil reserves will gradually reach the optimal level, and thereafter oil production can follow the optimal path.

Accordingly, a solution with an initial oil production which is higher than excisting capacity suggests that actual oil production should gradually increase above the optimal level until the optimal reallocation of uncertain assets have taken place.-

As a consequence of uncertainty, oil production in all alternatives is higher throughout the planning horizon than the projected oil production paths in the scenarios of the Perspective Analysis. However, in the alternatives of low risk aversion (2.1 and 3.2), optimal oil reserves are initially higher than the current estimate because it is more profitable to maintain uncertain assets. On the other hand, the optimal terminal oil reserves come out lower in all alternatives as compared to the projections of the Perspective Analysis. Thus the conclusion that uncertainty provides an incentive to increase oil production and convert the oil reserves into more certain assets, is near at hand. Our conclusion is in accordance with the present value calculations given in the report of the government appointed committee on the future extent of petroleum activities on the Norwegian continental shelf (NOU 1983:27, annex 14). In these present value considerations uncertainty was not explicitly taken into account. However, it must be kept in mind that although a lower risk aversion implies higher oil production, it also indicates that the initial oil reserves should be higher.

In terms of risk aversion, we have thus established alternatives for economic policy corresponding to what is considered by the Perspective Analysis as a relevant feasible set. Starting from the given stochastic assumptions and with a given degree of risk aversion, the stochastic optimization model yields guidelines for policy implications under uncertainty, which can be applied for evaluating long-term macroeconomic projections for the MSG-model.

The important lesson to be drawn is that moderate variations in the attitude toward risk may have fairly large impacts in terms of choices of economic policy.

REFERENCES

Aslaksen, I. and O. Bjerkholt (1984). Certainty equivalence procedures in the macroeconomic planning of an oil economy. Discussion paper, no.1. Central Bureau of Statistics.

Chow, G.C. (1975). Analysis and Control of Dynamic Economic Systems. John Wiley & Sons, New York. Chap. 8, pp. 197-201.

Johansen , L. (1960). A Multi-Sectoral Study of Economic Growth. North-Holland Publishing Company, Amsterdam.

Johansen, L. (1980). Parametric Certainty Equivalence procedures in decision-making under uncertainty. Zeitschrift für Nationalökonomie. Vol. 40, No. 3-4, pp. 257-279.

NOU 1983:27. The Future Extent of Petroleum Activities on the Norwegian Continental Shelf. Summary and conclusions of the Report from Commission appointed by the Royal Decree of 5 March 1982. Universitetsforlaget.

NOU 1983:37. Perspektivberegninger for norsk økonomi til år 2000. Universitetsforlaget.

Samuelson. P.A. (1969). Lifetime portfolio selection by dynamic stochastic programming. Review of Economics and Statistics, pp. 239-246.

Theil, H. (1964). Optimal Decision Rules for Government and Industry. North-Holland Publishing Company, Amsterdam.

ISSUED IN THE SERIES DISCUSSION PAPER

No. 1 I. Aslaksen and O. Bjerkholt: Certainty Equivalence Procedures in the Macroeconomic Planning of an Oil Economy.

No. 2 E. Offerdal: Structural Change, Profitability, and Survival.

No. 3 E. Biørn: On the Prediction of Population Totals from Sample Surveys Based on Rotating Panels.

No. 4 P. Frenger: A Short Run Dynamic Equilibrium Model of the Norwegian Production Sectors.

No. 5 I. Aslaksen and O. Bjerkholt: Certainty Equivalence Procedures in Decision-Making under Uncertainty: an Empirical Application.