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Hicksian Income from Stochastic Resource Rents

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Hicksian Income from Stochastic Resource Rents

Abstract:
The paper defines the risk adjusted Hicksian income as the highest consumption level that is consisten with utility being a martingale. We find that the appropriate risk adjustement is to compute wealth using a risk adjusted rate of return, but to compute the income as the risk free return to that wealth. The results are applied to estimation of Hicksian income from Norwegian petroleum wealth in the period 1973-1989.

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1 Introduction

According to Hicks famous definition, income is 'the amount a man can consume during a week and still be as well off at the end of the week as in the beginning.' The term 'as well off' has come to be interpreted as being able to maintain consumption. Thus income is the maximum sustainable consumption.

In his discussion of income, Hicks did not consider uncertainty. The purpose of this paper is to extend this definition of income to the case of uncertainty. Asheim and Brekke (1996) argues that a reasonable extension of the notion of sustainability under uncertainty is that the certainty equivalent of future consumption must be at least as high as current consumption. With expected utility, and under reasonable conditions this will imply that income is the maximum consumption, subject to the constraint that the consumption process must be feasible and a martingale in utility. We thus want to consider a case where income is generated from an exogenous resource rent process, and identify feasible martingale consumption processes.

For simplicity we assume that there is only one asset, and that it is risk-free with a constant return r. This is a strong assumption. The opposite extreme would be to assume complete markets, so we could construct a portfolio and a trading policy that replicated the resource rent process. The value of this portfolio would be the present value, calculated using a risk adjusted discount rate. By short selling the equivalent portfolio, we could capitalize the resource wealth and reinvest it in risk-free assets, and we can compute the Hicksian income from this wealth, without worrying about uncertainty in the consumption process. With a constant risk free rate of return, the income is the risk free return to wealth[1]. Note that in this case the present value of future resource rent will be computed

[1] Alternatively, we may choose to reinvest the wealth in risky assets, and apply the generalized definition of Hicksian income as used in this paper. That would be the suject of a separate study.

at a high risk adjusted discount rate, while the income, the return to the wealth, will be computed using a lower rate. In this paper we will prove that a similar risk adjustment procedure applies even with incomplete markets, where risky consumption cannot be avoided.

2 Risk neutrality

We first consider the case of risk neutrality. We then only require that the consumption process itself is a martingale, i.e. we require that $E\left[C_s|\mathcal{F}_t\right] \geq C_t$.

Consider an open economy, with income stream π_t and foreign assets b_t with a constant interest rate. Now

$$\dot{b}_t = rb_t + \pi_t - c_t$$

where c_t is consumption. Let $\mathcal{F}_t$ be the $\sigma-$algebra generated by the Brownian motion B_t. The resource rent is assumed to be $\mathcal{F}_t-$adapted. We further assume that $\pi_t \to 0$ as $t \to \infty$, and that a consumption process is feasible if $\lim_{t\to\infty} b_t e^{-rt} = 0$

Definition 1 *The resource wealth W_{Rt} is defined as*

$$W_{Rt} = E\left[\int_t^\infty \pi_s e^{-r(s-t)} ds|\mathcal{F}_t\right]$$

This definition of the resource wealth is consistent with the calculation of resource wealth in Aslaksen et al. (1990). This representation of the wealth , can be put on a differential form using the martingale representation theorem, see the proof of theorem 1. Thus

$$dW_{Rt} = \left(rW_{Rt} - \pi_t\right) dt + \phi(t,\omega) dB_t$$

for some process ϕ. For concreteness, consider the case where extraction Q_t is declining at a constant rate $Q_t = q\exp(-\lambda t)$, while the net price P_t is a geometrical Brownian motion

$$dP_t = \alpha P_t dt + \sigma P_t dB_t.$$

The rent is then $\pi_t = P_t Q_t$, and the present value of the rent, the wealth, is $W_{Rt} = P_t Q_t/(r+\lambda-\alpha)$. By Itô's lemma

$$dW_{Rt} = (rW_{Rt} - \pi_t)dt + \sigma W_{Rt} dB_t. \tag{1}$$

This observation will be useful in the following, but for now we only need the present value definition of wealth, and no further assumption about the revenues, except that it is adapted and that $\pi_t \to 0$ as $t \to \infty$.

Theorem 2 *The consumption process*

$$c_t = r\left(W_{R,t\wedge\tau} + b_{t\wedge\tau}\right), \ \textit{where } \tau = \inf\{t \geq 0 : W_{Rt} + b_t \leq 0\},$$

is the maximum feasible martingale consumption process.

The proof is in the appendix.

This result shows that with risk neutrality, the income generated by the resource rent process is the return to the resource wealth. Note that the interest rate r is used both to calculate the present value of the resource rent in the definition of wealth, and to compute the return. This is no longer true with risk aversion.

This conclusion seems quite intuitive. A risk neutral agent should not care whether his revenues are stochastic or not; only the expected revenues matters. We may then just as well replace the stochastic revenues with their expected value, and proceed as with full certainty.

3 Risk aversion

When we turn to risk aversion, the differential formulation of wealth, as in (6) is more convenient than present value definition. Hence we state the required assumptions in differential form.

Theorem 3 *Let*

$$db_t = (rb_t + \pi_t - c_t)dt$$

$$dW_{Rt} = (r_{Wt}W_{Rt} - \pi_t)dt + \sigma W_t dB_t$$

$$r_{Wt} = r + \frac{1}{2}\rho\sigma^2 \frac{W_{Rt}}{W_{Rt} + b_t}$$

with $b_0 = 0$, *and* $W_{Rt} = 0$ *for* $t \geq \tau$ *for some stopping time* τ. *Moreover, let*

$$c_t = r(b_t + W_{Rt})$$

and

$$U_t = \frac{1}{1-\rho} c_t^{1-\rho} \text{ with } \rho \geq 0$$

then U_t *is an* $\mathcal{F}_t-$*martingale.*

The proof is given in the appendix.

This theorem states that consuming the return to wealth is still a martingale, even with risk aversion, but the definition of resource wealth has changed. Comparing the differential form of W_{Rt} in with risk neutrality and risk aversion, we see that we use a higher discount rate with risk aversion. The difference $r_{Wt} - r$ is proportional to the degree of relative risk aversion.

There is a striking similarity between the resource wealth, W_{Rt} and the value of such a revenue stream with complete markets. In both cases the appropriate discount rates should be adjusted for risk. This implies more adjustment the farther into the future the revenues occur. This simply reflects that implicit in the diffusion term $\sigma W_t dB_t$ is an assumption that future revenues are much more uncertain that current revenues.

4 Back of the envelop calculation of risk adjusted Hicksian income

Unfortunately, the discount rate r_{Wt} varies stochastically with time, and hence it is difficult in practice to estimate the wealth. An approximation can be derived if we assume that $\frac{W_{Rt}}{W_{Rt}+b_t} \approx a$, where a is a constant. Then

$$r_W = r + \frac{1}{2}\rho a \sigma^2.$$

It seems reasonable to assume that the fluctuations in oil wealth is approximately equal to the fluctuation in net oil prices. Pindyck (1996) estimate $\sigma = 0.15 - 0.20$ for gross oil prices. This estimate is for the gross price, while the relevant price for the wealth is the net price, which currently is in order of magnitude about half the gross price. The volatility of the wealth should thus be higher than for the gross oil price, and we assume $\sigma = 0.35$. Calculations of National wealth SSB (1993), indicates that petroleum wealth accounts for only 5% of total wealth. If we consider the remaining part of national wealth as risk-free, this would correspond to $a = 0.05$, but it seems reasonable to assume that a considerable fraction of the remaining wealth is correlated to oil. For illustrative purpose, suppose 20% of main-land economic activity is perfectly correlated to the petroleum wealth, then $a \approx 0.25$. Assuming $\rho = 2$, and $r = 4\%$, we find that $r_W = 7.06\%$. Thus to make utility a martingale, we should compute the petroleum wealth with a discount rate at 7.06% and the return to the wealth at 4%. A similar calculation with risk aversion equal to $\rho = 5$,

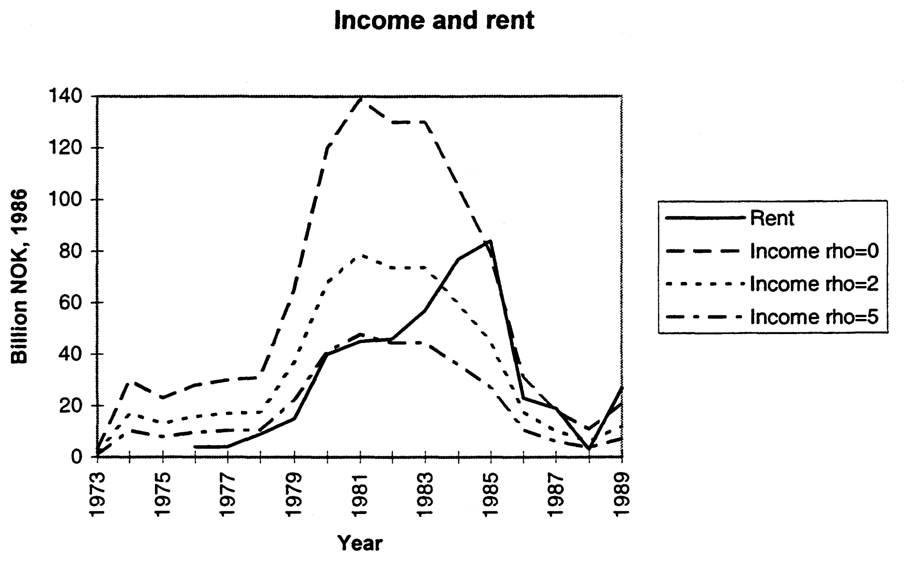

Figure 1: Risk-adjusted Hicksian Petroleum income.

we get $r_W = 11.6\%$.

In the calculations of Hicksian income from petroleum wealth in Aslaksen et al (1990), the same interest rate was used both to calculate the wealth as a present value and the return from the wealth. In that case the Hicksian income is not very sensitive to changes in rate of return, and the interest rate r_W would give about the same income as the 7% interest rate that was used. Calculating the return at the lower 4% rate would however reduce the income. With $\rho = 2$ the income would be reduced by almost 38%, while with $\rho = 5$, the reduction would be 60%. Based on Aslaksen et al (1990) we then get the following estimates of income and rent.

Note that even with moderate risk adjustment, the income is substantially higher than the rent in the early 1980's. Thus these estimates does not suggest that the nation should save during these years. With a considerable risk aversion however, the income is about at the level of rent until 1983, and thereafter revenues exceed the income.

To interpret the rate of risk aversion, we compute the risk premium for a consumption lottery that with equal probability increases or decreases consumption with 10%. With $\rho = 2$, we would be indifferent between this lottery and a certain 0.5% reduction in consumption, whereas with $\rho = 5$, we would be willing to accept 2.5% certain reduction in consumption to avoid the uncertainty.

A question that has been much discussed in Norway is to what extent the petroleum rent ought to be saved. The current model is not a normative model. To compute the income according to Hicks definition, is not to claim that the income is the amount that should be consumed. What the model does indicate, however, is how much the nation would need to save to be able to be able to maintain consumption in certainty equivalent.

The analysis has focused on the income from resource rent, but we assumed that a component of mainland revenues are perfectly correlated to the resource wealth. The adjustment compared to the income concept in Aslasken et al (1990) applies equally much to this component. Taking this into account we get the following development in required national saving.

5 Conclusion

We have presented a simple model that allows us to find the risk adjusted Hicksian income, and found that the appropriate risk aversion is to compute the wealth using a risk adjusted rate of return, and then using the risk free rate of return to compute income as the return to wealth. We have used this model to illustrate the importance of risk adjustment through a rough calibration of the core parameters. While Aslaksen et al (1990) found that income was much higher than the rent, we find that the risk adjusted income may be at the level of rents, with a substantial risk aversion.

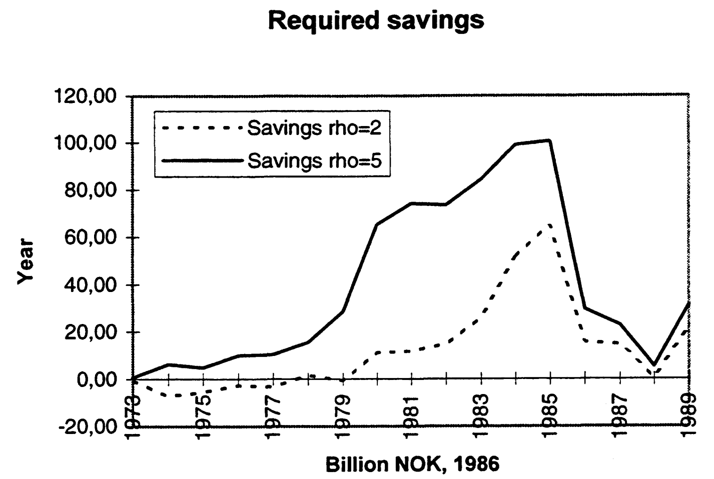

Figure 2: Savings required to maintain consumption

Note, however, that to achieve these conclusion we assumed that there is only one asset, the risk free one. This severely limits the possibility of diversification, and also the potential return to financial investments. Moreover, the calculations at this stage is rough back of the envelop calculations. Still the results indicate that risk adjusting the income measure may have a significant effect.

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A Proof of Theorem 1

Definition 4 *The real wealth is given by*

$$W_{R_t} = E[\int_t^\infty \pi_g e^{-r(s-t)} ds | \mathcal{F}_t] \tag{2}$$

Note that W_{Rt} is an $\mathcal{F}_t$-adapted process. Put

$$M_t(s) = E\left[\pi_g | \mathcal{F}_t\right]$$

then

$$W_{R_t} = \int_t^\infty M_t(s) e^{-r(s-t)} ds \tag{3}$$

This $\mathcal{F}_t$-adapted process $M_t(s)$, is a martingale. By the martingale representation theorem (Protter 1990), we know that

$$dM_t(s) = \phi(t, \omega; s) dB_t \tag{4}$$

for some process ϕ.

We differentiate (informally) W_{Rt} with respect to t, obtaining the formula

$$dW_{Rt} = -\pi_t + rW_{rt} + \int_t^\infty dM_t(s)e^{-r(s-t)}ds \tag{5}$$

This can be simplified to

$$dW_{Rt} = -\pi_t + rW_{rt} + \Phi(t,\omega)dB_t \tag{6}$$

where

$$\Phi(t,\omega) = -\int_t^\infty \phi(t,\omega;s)e^{-r(s-t)}ds \tag{7}$$

We state and prove this formally:

Lemma 5 *W_t as defined in (3) is the unique solution to the stochastic differential equation (6).*

Proof: We obtain, from formula (3),

$$
\begin{aligned}
W_{Rt} &= \int_t^\infty \left(M_0(s) + \int_0^t \phi(u,\omega;s)dB_u)e^{-r(s-t)}ds \right) \\
&= \int_t^\infty M_0(s)e^{-r(s-t)}ds + \int_t^\infty \int_0^t \phi(u,\omega;s)dB_u e^{-r(s-t)}ds \\
&= e^{-rt}W_{R_0} - \int_0^t M_0(s)e^{-r(s-t)}ds + \int_0^t \int_t^\infty \phi(u,\omega;s)e^{-r(s-t)}dsdB_u \\
&= e^{-rt}W_{R_0} - \int_0^t e^{-r(s-t)}\pi_s ds + \int_0^t \int_0^s \phi(u,\omega;s)e^{-r(s-t)}dB_u ds \\
&+ \int_0^t \int_t^\infty \phi(u,\omega;s)e^{-r(s-t)}dsdB_u \\
&= e^{rt}W_{R_0} - \int_0^t e^{-r(s-t)}\pi_s ds + \int_0^t \int_u^t \phi(u,\omega;s)e^{-r(s-t)}dsdB_u \\
&+ \int_0^t \int_t^\infty \phi(u,\omega;s)e^{-r(s-t)}dsdB_u \\
&= e^{rt}W_{R0} - \int_0^t e^{-r(s-t)}\pi_s ds + \int_0^t \phi(u,\omega)dB_u
\end{aligned}
$$

which is the unique solution of equation (6) (we have used the rule that we may change the order of integration). The uniqueness property follows by subtraction of candidates.

■

Definition 6 *The truncated total wealth is given by*

$$
W_t = b_{t\wedge\tau} + W_{Rt\wedge\tau} \tag{8}
$$

where τ is the first time $b_t = -W_{Rt}$, i.e. the nation is bankrupt.

It follows by adding formula 1.1 and 1.5 that

$$
\begin{aligned}
W_t &= W_0 + \int_0^{t\wedge\tau} [r(b_s + W_{Rs}) - c_s]\, ds + \int_0^{t\wedge\tau} \Phi(s,\omega)dB_s && (9) \\
&= W_0 + \int_0^t (rW_s) - X_{s\leq\tau}c_s)ds + \int_0^t X_{s\leq\tau}\Phi(s,\omega)dB_s && (10)
\end{aligned}
$$

We have used proposition 3.2.10 in (Karatzas and Shreve, 1991). It follows that if

$$c_t = r(b_t + W_{Rt}) \tag{11}$$

then

$$dW_t = X_{t<\tau}\Phi(t,\omega)dB_t \tag{12}$$

from which it follows that W_t is a $\mathcal{F}_t$-martingale. Note that the stopping time τ is now defined as the first time the process

$$W_0 + \int_0^t \Phi(s,\omega)dB_s \tag{13}$$

reaches zero. The modified consumption, defined as

$$C_t = c_{t\wedge\tau} = rW_t \tag{14}$$

is then also a $\mathcal{F}_t$-martingale. Which proves Theorem 1.

B Proof of theorem 2

Motivated from above section, we look at following system, with state variable (b_t, W_{Rt}),

$$\begin{aligned} db_t &= (rb_t + \pi_t - c_t)dt \\ dW_{Rt} &= (r_W W_{Rt} - \pi_t)dt + \sigma W_{Rt} dB_t \\ r_W &= r + \frac{1}{2}\rho\sigma^2 \frac{W_{Rt}}{(W_{Rt} + b_t)} \end{aligned}$$

with $b_0 = 0$ and $W_{Rs} = 0$ when $s > r$ some stopping time τ. Note that this formulation allows for a separate discount rate when computing the resource wealth.

Let now

$$c_t = rb_t + rW_{Rt} \tag{15}$$

and

$$U_t = u(c_t) \tag{16}$$

where u is the CRRA utility given by

$$u(x) = \frac{1}{1-\rho} x^{1-\rho} \text{ where } \rho \geq 0 \tag{17}$$

Then

$$\begin{aligned} dU_t &= \left(u'(c_t)(r_W - r)W_{Rt} + \frac{1}{2}u''(c_t)\sigma^2 W_{Rt}^2 \right) dt + u'(c_t) r\sigma W_{Rt} dB_t & (18) \\ &= u'(c_t) r\sigma W_{Rt} dB_t & (19) \end{aligned}$$

where we used that $xu''(x) = \rho u'(x)$. It follows that U_t is a $\mathcal{F}_t$-martingale.

Note: The existence of the above system seems to be hard to prove, not only because of the non-linearity, but also because of the very complicated boundary conditions.

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