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GROSS CAPITAL, NET CAPITAL, CAPITAL SERVICE PRICE, AND DEPRECIATION

A FRAMEWORK FOR EMPIRICAL ANALYSIS

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PREFACE

Time series of capital stocks play an important role in macroeconomic modelbuilding and analysis. They are also basic elements in the calculation of depreciation in the different production sectors for national accounting purposes.

This report presents a theoretical framework for the construction of capital stock figures from investment data. The results will be utilized in an empirical project which has been started recently in the Central Bureau of Statistics.

Central Bureau of Statistics, Oslo, 8 November 1983

Arne Øien

FORORD

Tidsserier for kapitalbeholdninger spiller en viktig rolle i makroøkonomisk modellbygging og analyse. I arbeidet med nasjonalregnskaper beregnes slike tidsserier blant annet som ledd i beregningen av kapitalslitet i de enkelte produksjonssektorer.

I denne rapporten presenteres et teoretisk opplegg for beregning av kapitaltall på basis av investeringsdata. Resultatene vil danne grunnlaget for et empirisk analyseprosjekt som nylig er satt i gang i Byråets forskningsavdeling.

Statistisk Sentralbyrå, Oslo, 8. november 1983

Arne Øien

CONTENTS

ABSTRACT

The construction of time series for capital stocks from data on gross investment is an essential element in the analysis of the firms' investment behaviour as well as in national accounting. In this report a general framework for the construction of such data is presented. Two capital concepts are involved - the gross capital - representing the capital's capacity dimension - and the net capital - representing its wealth dimension. The two associated concepts retirement (replacement) and depreciation are also dicussed, as is the formal relationship between the measurement of the capital volume and the measurement of the price of capital services. Finally, we propose and discuss some parametric survival profiles which may be useful in empirical applications.

1. INTRODUCTION*)

The measurement of real capital has been characterized as "one of the nastiest jobs that economists have set to statisticians" (John R. Hicks (1969, p. 253)). Closely related to it is the problem of measuring capital services, capital value, capital prices, capital service prices, and depreciation. The problem is not only one of measurement in the narrow statistical sense - a substantial part of the difficulty lies in the definition of useful concepts for empirical work. The reason for this lies in the fact that capital as an economic theoretical concept has at least two 'dimensions'. First, it is a capacity measure, a representation of the potential volume of capital services which can be 'produced' by the capital existing *at a given point of time*. Second, it is a wealth concept; capital has a value because of its ability to produce capital services *today and in the future*. The former concept is the one usually needed for production function studies, analyses of the firms' investment decisions, research on productivity issues, etc. The latter concept will be involved in analyzing the profitability of the production sectors, financial market studies, national accounting, etc. Obviously, both concepts have relevance to the building of large-scale macroeconomic models - *a priori*, there is, of course, nothing which implies that they should be numerically equal.

In this paper, we give a theoretical framework for constructing capital stock data (and data on related variables) from data on gross investment. Our approach will be a fairly general one, in that we work with generally specified survival profiles in all sections but one. Attention will be focused on two capital measures: the gross capital, which indicates the *instantaneous* productive capacity of the capital objects, and the net capital, which indicates their *prospective* capacity. Both variables can be constructed from previous investment data by applying two different, but related, weighting schemes. This is also the case for the two derived variables retirement - which is related to gross capital - and depreciation - which is based on net capital. The fifth variable with which we shall be concerned is the capital service price, which turns out to have a fairly close and empirically interesting relationship to the other variables.

The problems and concepts involved in the measurement of capital are, to some extent, equivalent to those encountered in demography. We may consider capital as a 'population' of capital units, associate investment with the 'birth' of a capital unit and retirement with 'death', etc. Demographic concepts as age, age distribution, survival probability, expected life time etc. are also useful when dealing with physical capital objects, and we shall make explicit reference to this equivalence at some places in the paper. There are, however, notable differences, especially when it comes to the definition of the wealth dimension of the capital stock, service prices, etc. Price variables, interest rates, and related concepts have, of course, no demographic counterparts.

The paper is organized as follows: In section 2, we introduce the concept survival function and give a formal definition of the variables gross capital and retirement (replacement). Two functions which are convenient for the following discussion are introduced in section 3. In section 4, we interpret the model probabilistically and show, *inter alia*, that the auxiliary functions introduced in section 3 are closely related to the moment generating function of the probability distribution of the capital's life time. Section 5 is concerned with the capital value and the associated variables net capital and depreciation. A corresponding definition of the capital service price is also given. In section 6, we take a closer look at the relationship between gross and net capital, depreciation and capital service price, both in the deterministic and stochastic interpretation of the model. Finally, in section 7, we present a selection of parametric specifications of the survival functions which may be useful in empirical applications. First, we consider the familiar exponential decay hypothesis - which has the remarkable property that gross capital and net capital coincide. Then we discuss four classes of two-parametric survival profiles, two of which are convex, two are concave, and some of their most interesting special cases.

In this paper, no attention will be devoted to the possible distortive effects of the corporate income tax system on the firm's investment decisions, through its impact on the capital service price. This issue is dealt with a related paper (Biørn (1983)), and we therefore disregard taxes altogether here.

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*) I wish to thank Petter Frenger and Øystein Olsen for their constructive comments on an earlier version of the paper, and Jørgen Ouren for his efficient programming of the computer routines.

2. THE GROSS CAPITAL: CAPITAL AS A CAPACITY CONCEPT. RETIREMENT (REPLACEMENT)

Let J(t) denote the quantity invested at time t, measured in physical units or as a quantity index[1]), where time is considered as continuous. More precisely, J(t) has the interpretation as the intensity of the investment flow at time t, and J(t)dt is the investment effectuated from time t to time t+dt. The proportion of an investment made s years (periods) ago which still exists as productive capital is denoted by B(s). The function B(s) represents both the physical wear and tear, and the time profile of the retirement of old capital goods. We shall consider it as a time invariant technical datum, in the following to be referred to as the *technical survival function.*

In principle, B(s) may be decomposed as

$$B(s) = B_S(s)B_E(s),$$

where $B_S(s)$ represents the relative number of capital units surviving at age s (the survival curve) and $B_E(s)$ indicates the efficiency of a capital unit of age s in relation to its efficiency at the time of investment, i.e. at age 0 (the efficiency factor). We shall not, however, make use of this decomposition in the following. We imagine that each capital good at each point of time contains a certain number of *'efficiency units'*, each having the same current productive capacity. The survival function B(s) indicates the relative number of efficiency units which are left s years after the initial investment was made. The function thus represents both the loss of efficiency of existing capital objects and physical disappearance, or retirement, of old capital goods. It is continuous and differentiable[2]) and has the following properties:

$$(1) \qquad 0 \leq B(s) \leq 1, \qquad \frac{dB(s)}{ds} \leq 0 \qquad \text{for all } s \leq 0,$$

$$B(0) = 1, \qquad \lim_{s\to\infty} B(s) = 0.$$

A typical survival function, with a finite maximal life time N, is illustrated in figure 1 below.

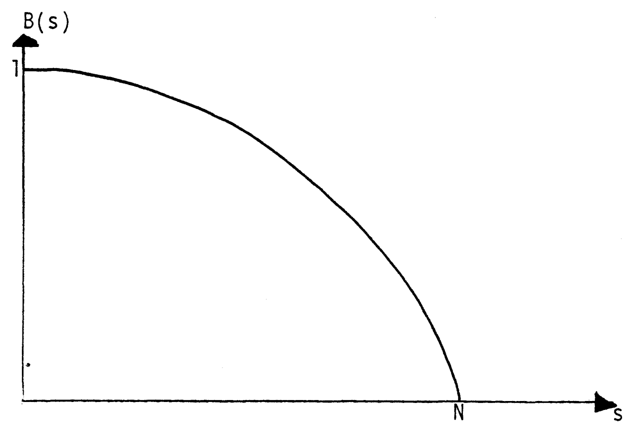

FIGURE 1. A typical curvature of the technical survival fuction B(s). N = maximal life time

1) Assuming that J(t) is an aggregate of homogeneous capital goods.
2) At least in the interior of the interval on which B(s) is strictly positive. Confer figure 1 and the examples given in section 7.

The service flow from this capital stock is an argument in a static production function, together with labour services and other inputs, and we assume throughout than the units of measurement and the form of the production function are chosen in such a way that *one capital (efficiency) unit produces one unit of capital services per unit of time.* Then

$$(2)\qquad K(t,s) = B(s)J(t-s) \qquad\qquad s \geq 0$$

has the double interpretation as the volume of the capital which is s years of age time t (i.e. the capital of vintage t-s existing at time t) and the service flow produced at time t by capital of age s.

Aggregation over capital vintages gives the following expression for the total volume of capital (flow of capital services) at time t:

$$(3)\qquad K(t) = \int_0^\infty K(t,s)ds = \int_0^\infty B(s)J(t-s)ds = \int_{-\infty}^t B(t-\theta)J(\theta)d\theta \qquad (\theta=t-s).$$

Capital thus defined is a *technical* concept; K(t) represents the *current productive capacity* of the total capital stock at time t. We shall refer to it as the *gross capital stock*. Differentiating (3) with respect to t we find that the rate of increase of the capital stock can be written as[3)]

$$(4)\qquad \dot{K}(t) = \frac{dK(t)}{dt} = B(0)J(t) + \int_{-\infty}^t \frac{dB(t-\theta)}{dt} J(\theta)d\theta = J(t) + \int_0^\infty \frac{dB(s)}{ds} J(t-s)ds$$

$$= J(t) - \int_0^\infty b(s)J(t-s)ds,$$

where

$$(5)\qquad \begin{cases} b(s) = -\dfrac{dB(s)}{ds}, \\ \text{which implies, since } B(0)=1, \\ B(s) = \displaystyle\int_s^\infty b(z)dz \qquad\qquad s \geq 0. \end{cases}$$

The volume of capital worn out or scrapped (i.e. the number of efficiency units which disappear) at time t is the difference between J(t), the gross investment, and the rate of increase of the (gross) capital stock. From (4) we find that the *volume of retirement* at time t can be expressed in terms of the previous investment flow as follows:

$$(6)\qquad D(t) = J(t) - \dot{K}(t) = \int_0^\infty b(s)J(t-s)ds.$$

We can alternatively call D(t) the volume of *replacement investment* at time t, since it represents the number of efficiency unit which would be required to replace retired equipment.

The function b(s) indicates the structure of the wear and tear and scrapping process: b(s)ds is the share of an initial investment of one unit which disappears from s to s+ds years after the time of installation. From (1) and (5) it follows that b(s) is non-negative for all s and that

$$(7)\qquad \int_0^\infty b(s)ds = 1.$$

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3) We utilize the following general formula for differentiating an integral:

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(t,\theta)d\theta = b'(t)f\{t,b(t)\} - a'(t)f\{t,a(t)\} + \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} f(t,\theta)d\theta.$$

This equation expresses the fact that all equipment installed will disappear sooner or later. We shall call b(s) *the (relative) retirement (replacement) function* in the sequel.

Formulae for gross capital and retirement similar to (3) and (6) can be found in e.g. Jorgenson (1974, pp. 191-192), and Hulten and Wykoff (1980, p. 100). The terminology, however, does not seem to be consistent in the literature. Some authors (e.g. Steele (1980)) define gross capital as the cumulated volume of past gross investment flow over a period of length N, the capital's life time, i.e. in our notation

$$K(t) = \int_0^N J(t-s)ds.$$

Others, e.g. Young and Musgrave (1980), use gross capital as synonymous with the capital measure derived from the perpetual inventory method, in stating that "gross capital stock for a given year [is obtained] by cumulating past investment and deducting the cumulated value of the investment that has been discarded". (Young and Musgrave (1980, pp. 23-24)). In our notation, this corresponds to

$$K(t) = \int_0^\infty B_S(s)J(t-s)ds.$$

This definition is also used by Johansen and Sørsveen (1967, p.182). It coincides with our definition (3) if $B(s)=B_S(s)$, which implies $B_E(s) = 1$ for all $s\geq 0$, i.e. if the efficiency of the surviving capital goods is the same for all vintages.4) If, moreover, $B_S(s) = 1$ for $0\leq s\leq N$, and 0 for $s>N$ - i.e. if all capital goods disappear simultaneously N years after investment - the three definitions of gross capital are equivalent. Our definition is the most general one, since it includes the others as special cases.

3. TWO USEFUL FUNCTIONS

To facilitate the following discussion, we introduce two auxiliary functions

$$(8)\qquad \phi_\rho(s) = \frac{\int_s^\infty e^{-\rho(z-s)}B(z)dz}{B(s)} = \frac{\int_0^\infty e^{-\rho\tau}B(\tau+s)d\tau}{B(s)} \qquad s\geq 0,$$

$$(9)\qquad \psi_\rho(s) = \frac{\int_s^\infty e^{-\rho(z-s)}b(z)dz}{B(s)} = \frac{\int_0^\infty e^{-\rho\tau}b(\tau+s)d\tau}{B(s)} \qquad s\geq 0,$$

where ρ is a positive constant, $\tau=z-s$, and B(s) and b(s) are defined as above. The numerator of $\phi_\rho(s)$ is the present value of the total flow of capital services produced by one initial unit of capital from the time it passes s years of age until it is scrapped, discounted to the time when it attains age s with a rate of discount equal to ρ. The denominator represents the share of the initial investment which attains age s.5) The ratio $\Phi_\rho(s)$ may thus be interpreted as *the discounted future service flow per capital (efficiency) unit which is s years of age.* Similarly, $\psi_\rho(s)$ has the interpretation as *the present value of the remaining retirement flow per capital unit which is s years of age.*

We then have in particular that

$$\phi_\rho(0) = \int_0^\infty e^{-\rho z}B(z)dz$$

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4) Our definition corresponds to the efficiency corrected capital stock as defined in section 4 of the Johansen-Sørsveen paper.

5) Or, more precisely, the relative number of efficiency units left s years after the time of investment.

is the present value of the total service flow from one *new* capital unit, and

$$\psi_\rho(0) = \int_0^\infty e^{-\rho z} b(z) dz$$

is the present value of the total replacement flow related to one *new* capital unit.

At this stage, however, it is not necessary to attach an economic interpretation to the functions $\Phi_\rho(s)$ and $\psi_\rho(s)$ and the parameter ρ; they may be considered as purely mathematical entities. Note, in particular, that we have said nothing so far about the possible relationship between ρ and economic market variables.

Obviously, $\Phi_\rho(s)$ and $\psi_\rho(s)$ are both decreasing functions of ρ for all values of s. From (5) and (9) it follows that

$$\psi_0(s) = 1 \qquad \text{for all s.}$$

If $\rho>0$, it is easy to show, by using integration by parts, that

$$(10) \qquad \Phi_\rho(s) = \frac{1}{\rho}\{1-\psi_\rho(s)\} \qquad s\geq 0,\ \rho>0.$$

All expressions which can be written in terms of $\Phi_\rho(s)$ can thus be written in terms of $\psi_\rho(s)$, and *vice versa.*

Differentiating (8) with respect to s, we find

$$(11) \qquad \Phi_\rho'(s) = \frac{b(s)}{B(s)} \int_0^\infty e^{-\rho\tau} \left\{ \frac{B(\tau+s)}{B(s)} - \frac{b(\tau+s)}{b(s)} \right\} d\tau.$$

This expression will be negative - i.e. $\Phi_\rho(s)$ is a decreasing function of s - if the integral in (11) is negative. Then $\psi_\rho(s)$ will be an increasing function of s, cf. (10). A sufficient condition for this to hold for all s, regardless of the value of ρ, is that

$$(12) \qquad \frac{B(\tau+s)}{B(s)} < \frac{b(\tau+s)}{b(s)} \qquad \text{for all s and } \tau>0.$$

In the next section, we give an interesting probabilistic interpretation of $\Phi_\rho(s)$ and $\psi_\rho(s)$.

4. A PROBABILISTIC INTERPRETATION

So far, we have considered the process generating the deterioration and retirement of the capital units as a deterministic process and we have established the functions B(s), b(s), $\Phi_\rho(s)$, and $\psi_\rho(s)$ on this basis. In this section, we shall give an alternative probabilistic interpretation and establish a correspondence between the two interpretations which will be useful for later reference.[6)]

When a capital good is installed, the investor does not normally know its actual life time. *Ex ante* it may be considered as a stochastic variable S, the function B(s) representing the *survival probabilities*, i.e. B(s) is the probability that a new capital good[7)] will survive for at least s years,

$$(13) \qquad B(s) = P(S\geq s) \qquad s\geq 0.$$

Since B(s) is continuous, the distribution function of the life time is

$$P(S\leq s) = 1 - B(s),$$

and b(s), as defined in (5), is the *density function of S,* since

6) When considering deterioration as a stochastic process, we take a step into 'renewal theory', a branch of mathematical statistics concerned with 'self renewing aggregates'. See Lotka (1939), Smith (1958), and Cox (1962).

7) Or more precisely, each of its efficiency units; cf. section 2.

$$b(s) = \frac{d}{ds}\{1-B(s)\} = -B'(s) \qquad s\geq 0.$$

The variable S represents the total life time of a capital good. Consider also the *remaining* life time of a capital good which has already attained age s, i.e. T=S-s. Using basic rules in probability calculus, we find that

$$(14) \qquad P(T\geq\tau|S\geq s) \equiv P(S\geq\tau+s|S\geq s) = \frac{B(\tau+s)}{B(s)} = B(\tau|s), \qquad s\geq 0,\ \tau\geq s,$$

where $B(\tau|s)$ is defined by the last equality. The *conditional density function of the remaining life time of capital which has attained age s* is thus

$$(15) \qquad b(\tau|s) \equiv -\frac{dB(\tau|s)}{d\tau} = \frac{b(\tau+s)}{B(s)} \qquad s\geq 0,\ \tau>0.$$

When this probabilistic interpretation of the retirement process is adopted, the share of a population of capital goods (efficiency units) which survive s years after investment will converge towards B(s) with a probability of one as the number of capital goods increases, according to the "law of the large numbers" - i.e. the former is a consistent estimator of the latter. Correspondingly, b(s)ds is (approximately) the proportion of the capital goods (efficiency units) whose life time is between s and s+ds years, and $b(0|s)ds = b(s)ds/B(s)$ represents the proportion of the capital goods having attained age s which will disappear before age s+ds. The latter is thus a formal analogue to the concept 'mortality rate' in demography, i.e. the probability that a person of a certain age will die during a given future period, e.g. the next year.

Which interpretations can then be given to the functions $\phi_\rho(s)$ and $\psi_\rho(s)$, defined in eqs. (8) and (9)? Let us first recall the definition of the concept *Laplace transform*. The Laplace transform of a stochastic variable X with a density function f(x), defined on $[0,\infty)$, is[8)]

$$(16) \qquad L_f(\lambda) = \int_0^\infty e^{-\lambda x}f(x)dx,$$

where λ is a parameter. Letting E denote the expectation operator, this is equivalent to

$$(17) \qquad L_f(\lambda) = E(e^{-\lambda X}).$$

Using (15), eq. (9) can be written as

$$(18) \qquad \psi_\rho(s) = \int_0^\infty e^{-\rho\tau}b(\tau|s)d\tau.$$

This is an expression of the form (16), with f(x) set equal to $b(\tau|s)$ and λ set equal to ρ. Thus $\psi_\rho(s)$ *stochastically interpreted is simply the Laplace transform of T=S-s, the remaining life time of a capital good which has attained age s.* This expression represented the present value of the remaining retirement flow per capital unit of age s in the deterministic interpretation of the model.

Eq. (18) can alternatively be written as (cf. (17))

$$(19) \qquad \psi_\rho(s) = E(e^{-\rho(S-s)}|S\geq s) = E(e^{-\rho T};\ s),$$

using ";s" as a shorthand notation for "$|S\geq s$". For s=0 we have in particular

$$(20) \qquad \psi_\rho(0) = L_b(\rho) = E(e^{-\rho S}),$$

8) See Feller (1966, Ch. XIII.1). The Laplace transform has a close relation to the moment generating function of the distribution. The moment generating function of X is simply $L_f(-\lambda) = E(e^{\lambda X})$. Confer Feller (1966, p. 411), or Cox (1962, p. 9).

i.e. $\psi_\rho(0)$ stochastically interpreted is the Laplace transform of the *total* life time of a new capital unit, S. Expanding $e^{-\rho T}$ in (19) by Taylor's formula, we obtain

$$(21) \qquad \psi_\rho(s) = E(1-\rho T + \frac{\rho^2 T^2}{2} - \frac{\rho^3 T^3}{6} \ldots ; s)$$

$$= 1 + \sum_{i=1}^{\infty} (-1)^i \frac{\rho^i}{i!} E(T^i; s), \qquad s \geq 0.$$

If we combine (21) and (10) we find

$$(22) \qquad \Phi_\rho(s) = E(T; s) + \sum_{i=2}^{\infty} (-1)^{i-1} \frac{\rho^{i-1}}{i!} E(T^i; s), \qquad s \geq 0.$$

By using this equation we can determine all the moments of the (conditional) distribution of the remaining life time T once we know the function $\Phi_\rho(s)$ for a value of ρ different from zero. All information about the distribution of T is thus "condensed" in this function. If $\rho=0$, the second term of (22) vanishes - i.e. all moments of second and higher order are "swept out" - and we get simply

$$(23) \qquad E(T; s) = \Phi_0(s) = \frac{1}{B(s)} \int_s^{\infty} B(z)dz.$$

For s=0, we have in particular

$$(24) \qquad E(S) = E(T;0) = \Phi_0(0) = \int_0^{\infty} B(z)dz.$$

Equations (23) and (24) reveal an interesting correspondence between the deterministic and the stochastic interpretation of the replacement process: What emerges as the *undiscounted* future service flow from one capital unit of age s in the deterministic framework[9)] is the expected remaining life time of a capital unit of age s in the stochastic version of the model, and *vice versa*. In particular, the total service flow from a new capital unit, deterministically interpreted, finds its counterpart in the expected total life time in the probabilistic interpretation.

5. THE NET CAPITAL: CAPITAL AS A WEALTH CONCEPT. CAPITAL SERVICE PRICE. DEPRECIATION

Gross capital as defined in section 2, by aggregating the surviving shares of the different capital vintages expressed in efficiency units, is a capacity concept: K(t) represents the number of capital (efficiency) units at time t on the one hand, and the instantaneous service flow from this capital stock on the other. We now consider the *value* dimension of the capital.

The market value of the capital goods will, in general, reflect the cost of producing new investment goods on the one hand, and the capital users' expectations about future productivity on the other. Let q(t) denote the price of investment goods at time t. The value of the investment outlay is then q(t)J(t), which is, of course, also the value of the new capital installed at time t. The value of an old capital good does not, in general, reflect its historic cost, but rather the service flow that it is likely to produce during its remaining life time. Let q(t,s) be the price of one capital unit (efficiency unit) of age s at time t and K(t,s), as before, the number of such units. The value of the capital which is of age s at time t is then V(t,s) = q(t,s)K(t,s), and the value of the total capital stock can be written as

9) Confer the interpretation of (8) above.

$$(25) \qquad V(t) = \int_0^\infty V(t,s)ds = \int_0^\infty q(t,s)K(t,s)ds = \int_0^\infty q(t,s)B(s)J(t-s)ds,$$

the last equality following from (2).

The decomposition of V(t,s) into a price and a quantity component is, however, in a sense arbitrary. An alternative decomposition is $V(t,s) = p(t,s)J(t-s)$, where $p(t,s) = q(t,s)B(s)$ has the interpretation as the price of capital of age s at time t per capital unit originally invested at time t-s.[10] The corresponding expression for the capital value,

$$(26) \qquad V(t) = \int_0^\infty V(t,s)ds = \int_0^\infty p(t,s)J(t-s)ds,$$

will be convenient for the purpose of defining depreciation, as we shall see in appendix A.

How is q(t,s), or p(t,s), determined? A reasonable assumption is that q(t,s) is an increasing function of the current investment price (the replacement price) q(t) for all s>0, and a decreasing function of the age s for each given t - the older a capital unit is, the lower will its price be, cet. par. Obviously, we have $q(t,0) = p(t,0) = q(t)$, and $V(t,0) = q(t)J(t)$.

In this paper, we shall make the specific assumption that the *relative* prices per unit of capital goods of different ages *perfectly* reflect the differences in their prospective service flows. More precisely, the price per unit of the (discounted) future flow of capital services is assumed to be the same for all capital vintages at each given point of time. Interpreting ρ as the rate at which future capital services are discounted (cf. section 3), we can formalize this hypothesis as

$$(27) \qquad \frac{q(t,s)}{\Phi_\rho(s)} = \frac{q(t)}{\Phi_\rho(0)} \qquad \text{for all } t \text{ and all } s \geq 0.$$

It implies a sort of '*law of indifference*' to hold between the different capital vintages: A firm buying at time t a capital unit (efficiency unit) of age s at the price q(t,s) pays the same price per unit of discounted prospective capital services as a firm which buys a new capital unit at the price q(t). If (27) is satisfied, the firm will be indifferent between expanding its capital stock by investing in new and old equipment, or by changing the age composition of the capital stock by investing in one vintage and disinvesting in another.[11] The common price per unit of (discounted) capital services is

$$(28) \qquad c(t) = \frac{q(t)}{\Phi_\rho(0)} = \frac{q(t)}{\int_0^\infty e^{-\rho s}B(s)ds} .$$

The 'law of indifference' (27) can alternatively be stated in terms of the price $p(t,s) = q(t,s)B(s)$. It then says

$$(29) \qquad \frac{p(t,s)}{\int_s^\infty e^{-\rho(z-s)}B(z)dz} = \frac{q(t)}{\int_0^\infty e^{-\rho z}B(z)dz} \qquad \text{for all } t \text{ and all } s \geq 0,$$

i.e. p(t,s), considered as a function of s, declines in equal proportion to the decline in the discounted remaining flow of capital services.

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10) A third decomposition would be the following: Let $B(s) = B_S(s)B_E(s)$, where $B_S(s)$ represents the survival curve and $B_E(s)$ the efficiency factor. We could then interpret $B_S(s)$ as belonging to the quantity component and $B_E(s)$ as belonging to the price component of V(t,s). The price variable,

$$q_E(t,s) = q(t,s)B_E(s) = p(t,s)/B_S(s),$$

would then represent the price per capital unit of age s at time t, corrected for loss of efficiency.

11) The latter conclusion, of course, presumes a neo-classical (putty-putty) production technology with full substitutability between the different capital vintages.

We may interpret ρ as the rate of interest forgone by a producer who owns the capital and uses its services instead of purchasing interest-bearing financial assets. If we set $\rho=r-\gamma$, where r is the nominal interest rate and γ is the rate of increase of q, and if r and γ are constants, then (28) is equivalent to

$$q(t) = \int_0^{\infty} e^{-rz} c(t+z)B(z)dz.$$

This equation agrees with the first-order conditions for maximization of the present value of cash-flow in a neo-classical model of producer's behaviour, when we replace c(t+z) by the value of the marginal productivity of capital at time t+z.[12)]

For the majority of capital goods, neither second hand markets nor hire markets exist, i.e. q(t,s) (or p(t,s)) and c(t) cannot be observed as market variables for s>0. The 'law of indifference' (27) - (29) is then no testable hypothesis; rather, it may be considered as providing an implicit definition of q(t,s) (or p(t,s)). It gives a procedure for *constructing* series for q(t,s), and corresponding indices for c(t), under perfect market conditions, from observed values of the investment price q(t) and given values of the survival rates B(s) and the rate of discount ρ.[13)]

Returning for a moment to the probabilistic interpretation of the deterioration process, we find, by using (22), that (27) can be expressed in terms of the moments of the distribution of the capital's life time as follows

$$(30)\qquad \frac{q(t,s)}{E(T;s) + \sum_{i=2}^{\infty}(-1)^{i-1}\frac{\rho^{i-1}}{i!}E(T^i;s)} = \frac{q(t)}{E(S) + \sum_{i=2}^{\infty}(-1)^{i-1}\frac{\rho^{i-1}}{i!}E(S^i)} \qquad \text{for all } t \text{ and all } s\geq 0.$$

This equation has particular intuitive appeal in the case where the discounting rate ρ is zero. The 'law of indifference' then simply says that *the relative prices of the different capital vintages are equal to the ratios of their expected remaining life times:*

$$\frac{q(t,s)}{q(t)} = \frac{E(T;s)}{E(S)} \qquad \text{for all } t \text{ and all } s\geq 0,\ \rho=0.$$

Combining (25), (27), and (28), we find that the value of the capital stock can be written as

$$(31)\qquad V(t) = q(t)\int_0^{\infty}\frac{\Phi_\rho(s)B(s)}{\Phi_\rho(0)}J(t-s)ds = c(t)\int_0^{\infty}\Phi_\rho(s)B(s)J(t-s)ds.$$

This equation gives a procedure for computing the capital value from data on q(t), J(t-s), B(s), and ρ. It also indicates two alternative ways of decomposing this value into a price and a quantity component.

First, if we define the price component as equal to the *current investment price*, the quantity component becomes

$$(32)\qquad K_N(t) = \frac{V(t)}{q(t)} = \frac{1}{q(t)}\int_0^{\infty} q(t,s)K(t,s)ds = \int_0^{\infty} G_\rho(s)J(t-s)ds,$$

where

$$(33)\qquad G_\rho(s) = \frac{\Phi_\rho(s)B(s)}{\Phi_\rho(0)} = \frac{\int_s^{\infty} e^{-\rho(z-s)}B(z)dz}{\int_0^{\infty} e^{-\rho z}B(z)dz} \qquad s\geq 0.$$

12) See Biørn (1983, appendix) for a demonstration of this in a more general context.
13) In the rather few cases where q(t,s) (or p(t,s)) are observed market variables - e.g. cars, office buildings, and dwellings - eq. (27) (or (29)) can be used to estimate $\Phi_\rho(s)$ and hence, given the rate of discount ρ, draw conclusions on the form of the underlying survival function B(s). Examples of analyses of this sort are Hall (1971) and Hulten and Wykoff (1981).

We see that $K_N(t)$, like $K(t)$, is constructed by aggregating the previous investment flow, but the weighting system is basically different. The weight assigned to investment made s years ago in $K_N(t)$, $G_\rho(s)$, is the share of the total discounted service flow produced by one unit invested *after it is s years old*, whereas $K(t)$ is based on the technical survival rates $B(s)$. Or othervise stated, $K_N(t)$ is constructed on the basis of the *prospective* service flow, $K(t)$ on the basis of the *instantaneous* service flow each capital vintage. From (33) we see that the weighting function $G_\rho(s)$ satisfies

$$(34)\qquad 0 \le G_\rho(s) \le 1, \qquad \frac{dG_\rho(s)}{ds} \le 0 \qquad \text{for all } s \ge 0,$$

$$G_\rho(0) = 1, \qquad \lim_{s\to\infty} G_\rho(s) = 0,$$

i.e. it has the same qualitative properties as $B(s)$, cf. (1). Furthermore, it follows from (11) and (33) that $G_\rho(s)<B(s)$ for all s if the inequality (12) is satisfied for all s. This is thus a sufficient condition for $K_N(t)<K(t)$ to hold for all t, irrespective of the discounting rate ρ and of the time profile of the investment. We shall refer to $K_N(t)$ as the *net capital stock* in the following.[14)]

Second, if we decompose $V(t)$ by setting its price component equal to the *price per unit of capital services*, as defined in (28), we get a quantity component equal to

$$(35)\qquad K_S(t) = \frac{V(t)}{c(t)} = \int_0^\infty \Phi_\rho(s)B(s)J(t-s)ds$$

$$= \int_0^\infty \{\int_s^\infty e^{-\rho(z-s)}B(z)dz\}J(t-s)ds = K_N(t)\cdot\Phi_\rho(0).$$

Since $\Phi_\rho(s)B(s)J(t-s) = \{\int_s^\infty e^{-\rho(z-s)}B(z)dz\}J(t-s)$ is the present value of the remaining service flow from capital vintage t-s, $K_S(t)$ has the interpretation as the present value of the total future service flow from the capital stock existing at time t.

With these definitions, we thus get the following simple and attractive relationship between the capital value, the investment price, the price of capital services, the capital volume, and the volume of capital services:

$$(36)\qquad V(t) = q(t)K_N(t) = c(t)K_S(t),$$

or

Value of capital stock
= Current investment price x Volume of net capital stock
= Current capital service price
x Volume of (discounted) future services from existing capital stock.

14) This term is used to some extent in the literature on the measurement of capital, but its precise meaning is not always made clear, and it seems to be some differences in terminology. Often, the concept is defined by general statements like "Net capital stock is obtained .... by deducting [from the cumulated past investment] the cumulated value of depreciation" (Young and Musgrave (1980, p. 24)), and "Gross capital stock, less the amount of accrued capital consumption gives net capital stock. Net capital stock is two dimensional in that it reflects not only the amount of capital in current use, but also the unexpired future potential of those assets" (Steele (1980, p. 227)). This usage is consistent with our definition in eq. (32) in some cases, but not in others.

We now turn to the concept *depreciation*. It has the same formal relationship to the net capital stock as the concept retirement, defined in section 2, has to the gross capital stock. Depreciation can, however, be expressed both in value and volume terms. We define the (net) *value* of depreciation as the difference between the current investment expenditure and the increase in the capital value, i.e.

(37) $E(t) = q(t)J(t) - \dot{V}(t).$

Likewise, the *volume* of depreciation is, by definition, the difference between the current investment quantity and the increase in the net capital stock:

(38) $D_N(t) = J(t) - \dot{K}_N(t).$

From (36) - (38) we obtain

(39) $$E(t) = q(t)\,[J(t) - \dot{K}_N(t)] - \dot{q}(t)K_N(t)$$

$$= q(t)D_N(t) - \dot{q}(t)K_N(t),$$

i.e. the following accounting relationship exists between depreciation in value and volume terms:

(Net) value of depreciation

= *Investment price x Volume of depreciation*

− *Increase in investment price x Volume of net capital stock.*

Interpreting $q(t)D_N(t)$ as the *gross* value of depreciation and $\dot{q}(t)K_N(t)$ as the value of the *appreciation* (capital gains), we can alternatively state this relationship as

(Net) value of depreciation

= *Gross value of depreciation*

− *Value of appreciation.*

We can express $D_N(t)$ and $E(t)$ in terms of the previous investment flow $J(t-s)$. From (32) we obtain[15)]

(40) $$\dot{K}_N(t) = J(t) - \int_0^\infty g_\rho(s)J(t-s)ds,$$

where

(41) $$g_\rho(s) = -\frac{dG_\rho(s)}{ds} = \frac{B(s)}{\Phi_\rho(0)}\{1-\rho\Phi_\rho(s)\} = \frac{B(s)\psi_\rho(s)}{\Phi_\rho(0)} = \frac{\int_s^\infty e^{-\rho(z-s)}b(z)dz}{\int_0^\infty e^{-\rho z}B(z)dz},$$

the last three equalities following successively from (33), (10), (9), and (8). Hence,

(42) $$D_N(t) = \int_0^\infty g_\rho(s)J(t-s)ds = \int_0^\infty \frac{B(s)}{\Phi_\rho(0)}\{1-\rho\Phi_\rho(s)\}J(t-s)ds = \int_0^\infty \frac{\psi_\rho(s)}{\Phi_\rho(0)}B(s)J(t-s)ds.$$

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15) Confer the formally similar derivation of (4) above.

Eqs. (39), (32), and (42) then give

$$(43)\qquad E(t) = q(t) \int_0^\infty \{g_\rho(s) - \frac{\dot{q}(t)}{q(t)} G_\rho(s)\} J(t-s)ds$$

$$= q(t) \int_0^\infty \frac{B(s)}{\Phi_\rho(0)} \{1-(\rho+ \frac{\dot{q}(t)}{q(t)}) \Phi_\rho(s)\} J(t-s)ds.$$

Equations (42) and (43) indicate a procedure for calculating depreciation in volume and value terms which is consistent with (3), (32), and (6) for gross capital, net capital, and replacement.

The function $g_\rho(s)$, as defined in (41), represents the structure of the depreciation, in the same way as b(s) represents the retirement process. In particular, $g_\rho(s)$, like b(s), may be given a probability density interpretation since it is non-negative with

$$(44)\qquad \int_0^\infty g_\rho(s)ds = 1.$$

Literally, depreciation means 'decline in value (or decline in price)'. Hence, it may be argued that this variable should be defined on the basis of the price component of the capital stock, not as a quantity concept, as (38) and (42) implies. In appendix A, we interpret depreciation in terms of the vintage prices p(t,s), and show that this interpretation is equivalent to the quantity interpretation given above. It represents an alternative way of decomposing the value E(t).

6. THE RELATIONSHIP BETWEEN DEPRECIATION, GROSS CAPITAL, NET CAPITAL, AND CAPITAL SERVICE PRICE - FURTHER RESULTS

There exist other relationships between the variables we have introduced in the previous sections which are worth noting. We shall call attention to a few of them.

From (3), (32), (33), and (42) it follows that depreciation, net capital, and gross capital satisfy the following equation

$$(45)\qquad D_N(t) + \rho K_N(t) = \frac{K(t)}{\Phi_\rho(0)}.$$

Recalling our definitions of these three variables, this is a remarkably simple relationship. It can, for instance, be used in combination with (6) and (38) to facilitate the computation of gross and net capital from investment data - or to check the consistency of the resulting series. Furthermore, combining (28) and (45), we find

$$(46)\qquad q(t)D_N(t) + \rho q(t)K_N(t) = c(t)K(t).$$

If we interpret ρ as an interest rate and $\rho q(t)K_N(t) = \rho V(t)$ as the implicit interest cost on the capital value, this equation says that

(Gross) value of depreciation + Interest on capital value
= Capital service price x Volume of gross capital stock.

It gives, in other words, two alternative ways of expressing the current 'user value' of the capital stock.

If the interest rate applied in discounting the future capital services, ρ, is zero, (45) becomes simply

(47) $$D_N(t) = \frac{K(t)}{\Phi_0(0)} = \frac{\int_0^\infty B(s)J(t-s)ds}{\int_0^\infty B(s)ds} \qquad (\rho=0).$$

In this case, the depreciation is proportional to the gross capital stock, the factor of proportionality being the inverse of the total service flow from one capital unit during its life time.

When $\rho=0$, there are also interesting probabilistic analogues to the deterministic interpretations given above. First, from (30) and (32) it follows that the volume of the net capital can be written as

$$K_N(t) = \int_0^\infty \frac{E(S-s;s)}{E(S)} K(t,s)ds \qquad (\rho=0).$$

It thus emerges as a weighted sum of the remaining part of each capital vintage, the weights being the expected remaining life time as a fraction of the total life time. Gross capital is the corresponding unweighted sum

$$K(t) = \int_0^\infty K(t,s)ds.$$

Interpreting the model in "demographic" terms, (while disregarding differences in efficiency) we might thus say that the measurement of gross capital finds its counterpart in a traditional population census, whereas the measurement of the net capital corresponds to a fictitious population census in which each person is given a weight equal to his expected remaining life time as estimated from life tables. (If $\rho > 0$, higher order moments of the distribution of the life time should also be taken into consideration, cf. (30), and the comparison loses some of its intuitive appeal.)

Second, since $E(S) = \Phi_0(0)$ (cf. (24)), the expressions for the depreciation given in (47) can be interpreted stochastically as

(48) $$D_N(t) = \frac{K(t)}{E(S)} = \frac{\int_0^\infty P(S\geq s)J(t-s)ds}{\int_0^\infty P(S\geq s)ds} \qquad (\rho=0),$$

i.e. depreciation is equal to gross capital divided by the expected life time of a new capital unit (first equality), or equivalently, equal to a weighted average of the past gross investment flow with the survival probabilities $B(s) = P(S\geq s)$ used as weights (second equality). We get a similar relationship between the price variables. From (23), (24), (27), and (28), we find

(49) $$c(t) = \frac{q(t)}{E(S)} = \frac{q(t,s)}{E(S-s;s)} \qquad \text{for all } s \ (\rho=0),$$

i.e. the capital service price is equal to the market price per capital unit divided by its expected (remaining) life time. And this equality holds for all capital vintages. (Again, when $\rho>0$, higher order moments should also be taken into account.)

Third, as we noticed above (eqs. (41) and (44)) $g_\rho(s)$ has properties which suggest its interpretation as a density function. This function has the same formal relationship to the net capital $K_N(t)$ as the function $b(s)$ has to the gross capital $K(t)$. Since $b(s)$, interpreted stochastically, is the density function of the life time of the *gross* capital, S, this motivates giving $g_\rho(s)$ the interpretation as the density function of the 'life time of *net* capital', S_N. The formal definition of S_N would then be

$$P(S_N\geq s) = G_\rho(s) \qquad \text{for all } s\geq 0.$$

Using (41), we find that its expectation is in general

$$(50)\qquad E(S_N) = \int_0^\infty s g_\rho(s)ds = \frac{\int_0^\infty sB(s)\psi_\rho(s)ds}{\Phi_\rho(0)} .$$

For $\rho=0$ we get in particular

$$(51)\qquad E(S_N) = \int_0^\infty s\, g_0(s)ds = \frac{\int_0^\infty sB(s)ds}{\int_0^\infty B(s)ds} \qquad (\rho=0).$$

Thus defined, the expected life time of the net capital would then emerge as a weighted average of the life time with the survival probabilities used as weights.

The latter equation can be given an interesting reformulation. Using integration by parts, it is easy to show that $\int_0^\infty sB(s)ds = E(S^2)/2$, provided that $\lim_{s\to\infty} s^2B(s) = 0$. Hence, recalling (24), we find

$$(52)\qquad \frac{E(S_N)}{E(S)} = \frac{1}{2}\frac{E(S^2)}{[E(S)]^2} = \frac{1}{2}\frac{[E(S)]^2+\sigma_S{}^2}{[E(S)]^2} = \frac{1}{2}\{1+[\frac{\sigma_S}{E(S)}]^2\} \qquad (\rho=0),$$

where σ_S^2 is the variance of S. The ratio between the expected life time of the net capital as defined above and that of the gross capital thus has its lowest value, 1/2, for $\sigma_S=0$, i.e. when there is no uncertainty with respect to the life time of the gross capital; all units disappear at the same time. The ratio increases with the square of the coefficient of variation of the life time, $\sigma_S/E(S)$. If the coefficient of variation is unity, the expected life time of gross and net capital coincide.

7. PARAMETRIC SURVIVAL FUNCTIONS

The results derived in the previous sections are valid for any survival function B(s) which satisfies the general restrictions (1). In this section, we present a selection of parametric functions which may be useful for empirical applications. For each B(s) we derive the corresponding functions $G_\rho(s)$, $\Phi_\rho(s)$, and $\psi_\rho(s)$. These functions can be used on the one hand for the quantification of gross and net capital, retirement, depreciation, and capital service price on the basis of investment data - on the other hand for estimating and testing hypotheses about the form of the survival function from data on vintage prices.

We present *four classes* of survival functions, each characterized by *two parameters*. The first parameter represents the maximal life time of the capital, the second indicates the 'curvature' of the survival profile. Important special cases of these functions are also considered. The results will be presented partly algebraically, and partly in the form of tables and diagrams. For the sake of reference we shall, however, start by considering a one parameter survival function, namely the familiar specification with exponentially declining survival rates.

Exponentially declining survival function: $B(s) = e^{-\delta s}$

Consider the parametrization

$$(53)\qquad B(s) = e^{-\delta s} \qquad s\geq 0,$$

where δ is a positive constant. Probabilistically interpreted, the life time S then has an exponential distribution. Inserting (53) in (5), (8), and (9), we find

(54) $b(s) = \delta e^{-\delta s}$,

(55) $\Phi_\rho(s) = \frac{1}{\rho+\delta}$,

(56) $\psi_\rho(s) = \frac{\delta}{\rho+\delta}$ $s \geq 0$.

This parametrization thus has the particular property that $\Phi_\rho(s)$ *and* $\psi_\rho(s)$ *are constants independent of s*. The (conditional) Laplace transform of the remaining life time is equal to the Laplace transform of the total life time for all ages s. Since

$$\frac{1}{\rho+\delta} = \frac{1}{\delta}[1 - \frac{\rho}{\delta} + (\frac{\rho}{\delta})^2 - \ldots\ldots],$$

we find, by using (22), that

(57) $E(T;s) = E(S) = \frac{1}{\delta}$,

$E(T^2;s) = E(S^2) = \frac{2}{\delta^2}$, for all $s \geq 0$,

and hence

(58) $\mathrm{var}(T^2;s) = \mathrm{var}\ (S^2) = E(S^2) - [E(S)]^2 = \frac{1}{\delta^2}$ for all $s \geq 0$.

In this case, the remaining life time has a (conditional) expectation equal to $1/\delta$ and a (conditional) variance equal to $1/\delta^2$ for all s.

From (33), (41), and (55) we find moreover that

(59) $G_\rho(s) = B(s) = e^{-\delta s}$,

(60) $g_\rho(s) = b(s) = \delta e^{-\delta s}$ for $s \geq 0$,

and hence, using (32) and (42), that

(61) $K_N(t) = K(t) = \int_0^\infty e^{-\delta s} J(t-s)ds$,

(62) $D_N(t) = D(t) = \int_0^\infty \delta e^{-\delta s} J(t-s)ds = \delta K_N(t) = \delta K(t)$.

These relationships hold regardless of the value of the discounting rate ρ. Thus, *in the exponential case, gross capital is numerically equal to net capital, and retirement (replacement) coincides with depreciation.*[16] The rate of retirement is equal to the rate of depreciation, and the common value is constant and equal to δ. This is another particular property of this survival function.

Its implication for the price variables is also remarkably simple. From (27) and (55) it follows that

(63) $q(t,s) = q(t)$ for all $s \geq 0$,

i.e. the price per capital *efficiency* unit will be the same for all ages. The equivalent relationship expressed in terms of the price per capital unit *originally* invested is

16) This conclusion concurs with eq. (52) which implies that $E(S_N) = E(S)$ when the coefficient of variation of S is unity. This is in fact the case for the exponential distribution, since the expectation and the standard deviation are both equal to $1/\delta$ in this case, cf. (57) and (58).

(64) $p(t,s) = e^{-\delta s}q(t),$ for all $s \geq 0$,

i.e. this price declines exponentially with age at the rate δ. Combining (28) and (55) we find that the capital service price is equal to

(65) $c(t) = q(t)(\rho+\delta)$.

If we let $\rho = r - \dot{q}(t)/q(t)$, r denoting the nominal market interest rate, i.e. if we make the reasonable equilibrium assumption that the capital users (capital owners) consider the current 'real interest rate' when discounting the future flow of capital services from time t (confer section 5), this expression is identical with the familiar textbook formula for the user cost of capital in a neo-classical model of capital accumulation,

$$c(t) = q(t)\{r+\delta-\dot{q}(t)/q(t)\} \; .$$

From the point of view of empirical applications, the exponential model is very restrictive since it has only one parameter. Its implicit assumption of an infinite maximal service life is also inconvenient and implausible, as is the constancy of the rate of depreciation which it imposes. In the following, we outline four classes of two-parametric survival functions with a finite maximal life time, two of which are convex and two concave.

CLASS I: Convex: $B(s) = (1-\frac{s}{N})^n$.

First we consider

(66) $$B(s) = B^I(s;N,n) = \begin{cases} (1-\frac{s}{N})^n & \text{for } 0 \leq s \leq N \\ 0 & \text{for } s>N, \end{cases}$$

where N and n are positive[17] constants, n integer. The corresponding retirement (density) function is

(67) $$b(s) = b^I(s;N,n) = \begin{cases} \frac{n}{N}(1-\frac{s}{N})^{n-1} & \text{for } 0 \leq s \leq N \\ 0 & \text{for } s>N. \end{cases}$$

It is convenient to introduce the auxiliary function

(68) $$C_\rho^I(s;N,n) = \int_s^N e^{-\rho(z-s)}(1-\frac{z}{N})^n dz \qquad 0 \leq s \leq N.$$

When $\rho=0$, integration yields directly

(69) $$C_0^I(s;N,n) = \frac{N}{n+1}(1-\frac{s}{N})^{n+1} = \frac{N}{n+1}B^I(s;N,n+1) \qquad 0 \leq s \leq N.$$

If $\rho > 0$, the function satisfies the following recurrence formula, proved in appendix B:

17) Formally, B^I is also defined for n=0, but not b^I; confer below.

$$(70)\qquad C_\rho^I(s;N,n) = \frac{1}{\rho}[(1-\frac{s}{N})^n - \frac{n}{N}\,C_\rho^I(s;N,n-1)]$$

$$= \frac{1}{\rho}[B^I(s;N,n) - \frac{n}{N}\,C_\rho^I(s;N,n-1)] \qquad n=1,2,\ldots,$$

$$C_\rho^I(s;N,0) = H_\rho(N-s) = \frac{1}{\rho}[1-e^{-\rho(N-s)}],$$

where, in general, $H_a(M)$ denotes the present value of a constant annuity of 1 discounted over M years at the rate a.

Inserting (66) and (67) in (8) and (9), it follows that Φ_ρ and ψ_ρ can be expressed as

$$(71)\qquad \Phi_\rho(s) = \frac{C_\rho^I(s;N,n)}{B^I(s;N,n)} \qquad 0\leq s\leq N,$$

$$(72)\qquad \psi_\rho(s) = \frac{n}{N}\,\frac{C_\rho^I(s;N,n-1)}{B^I(s;N,n)} \qquad 0\leq s\leq N.$$

Hence, using (33) and (41), we find that the weighting functions for net capital and depreciation are, respectively,

$$(73)\qquad G_\rho(s) = \frac{C_\rho^I(s;N,n)}{C_\rho^I(0;N,n)},$$

$$(74)\qquad g_\rho(s) = \frac{n}{N}\,\frac{C_\rho^I(s;N,n-1)}{C_\rho^I(0;N,n)}.$$

For $\rho=0$ we get in particular

$$G_0(s) = (1-\frac{s}{N})^{n+1} = B^I(s;N,n+1),$$

$$g_0(s) = \frac{n+1}{N}(1-\frac{s}{N})^n = b^I(s;N,n+1).$$

When no discounting of future capital services is performed, there is thus a very simple relationship between the weighting function of the gross capital and that of the net capital in this case: We only have to change n to n+1 to get from the former to the latter.

From (23), (66), (69), and (71) we find

$$(75)\qquad E(T;s) = \Phi_0(s) = \frac{N-s}{n+1},$$

i.e. the expected remaining life time is a linearly decreasing function of age, with a rate of decrease equal to $1/(n+1)$. In particular, the expected life time of a new capital unit is

$$E(S) = \frac{N}{n+1}.$$

The corresponding 'expected life time of net capital', as defined in (51), is

$$E(S_N) = \frac{N}{n+2} \qquad (\rho=0),$$

and hence their ratio

$$(76) \qquad \frac{E(S_N)}{E(S)} = \frac{n+1}{n+2} \qquad (\rho=0)$$

is less than one for all admissible values of n and increases to one with increasing n, regardless of the value of N.

The expressions for the price variables follow by substituting (71) in (27), (29), and (28). For the vintage prices we get

$$(77) \qquad q(t,s) = q(t)\,\frac{C_\rho^I(s;N,n)}{C_\rho^I(0;N,n)(1-\frac{s}{N})^n},$$

$$(78) \qquad p(t,s) = q(t)\cdot\frac{C_\rho^I(s;N,n)}{C_\rho^I(0;N,n)},$$

and the capital service price becomes

$$(79) \qquad c(t) = \frac{q(t)}{C_\rho^I(0;N,n)}.$$

In these expressions, the effect of the interest rate ρ on the one hand, and the parameters characterizing the survival profile, N and n, on the other, are intermingled. Since $C_\rho^I(0;N,n)$ is a highly nonlinear function of ρ, N, and n, we cannot, for instance, decompose the capital service price into two additive components, one representing 'interest cost' and the other representing 'depreciation'. This is an important difference between this survival function and the exponential one, which admits an additive decomposition; cf. (65). We can, however, find the isolated effect of the depreciation component by setting $\rho=0$ in (77)-(79). This gives

$$q(t,s) = q(t)(1-\frac{s}{N}),$$

$$p(t,s) = q(t)(1-\frac{s}{N})^{n+1} \qquad 0\le s\le N,$$

$$c(t) = q(t)\frac{n+1}{N}.$$

In this degenerate case, the capital service price is inversely proportional to the maximal life time N and proportional to n+1.

The class of survival profiles (66) contains several specifications discussed in the literature as special class. Let us look briefly at a few of them.

n=0: Simultaneous exit ("One horse shay")

In this case, all capital objects are assumed to retain their full productive capacity during N periods and are then completely scrapped. Probabilistically interpreted, the distribution of the technical life time S is a 'one point distribution'; the entire 'probability mass' is concentrated in the point s=N. From (70), (76), (77), and (78) we find

$$q(t,s) = p(t,s) = q(t) \cdot \frac{1-e^{-\rho(N-s)}}{1-e^{-\rho N}} = q(t)\,\frac{H_\rho(N-s)}{H_\rho(N)} ,$$

$$c(t) = q(t) \cdot \frac{\rho}{1-e^{-\rho N}} = \frac{q(t)}{H_\rho(N)} .$$

The latter is the familiar formula

$$\text{user cost} = \frac{\text{investment price}}{\text{annuity factor}} .$$

Eq. (73) gives in this case, when we use de L'Hôpital's rule,

$$G_\rho(s) = \frac{1-e^{-\rho(N-s)}}{1-e^{-\rho N}} = \xrightarrow[\rho\to 0]{} 1 - \frac{s}{N} ,$$

and hence

$$g_\rho(s) = \frac{\rho e^{-\rho(N-s)}}{1-e^{-\rho N}} \xrightarrow[\rho\to 0]{} \frac{1}{N} .$$

Finally, when n=0, (76) gives

$$\frac{E(S_N)}{E(S)} = \frac{1}{2} \qquad (\rho=0),$$

which agrees with (52'), since the simultaneous exit specification implies, as already remarked, that the life time S has a one point distribution, and, consequently, $\sigma_S=0$. The ratio of $E(S_N)$ and $E(S)$ cannot take a lower value than it does in this case, so the simultaneous exit assumption is also in this respect an extreme specification.

n=1: Linear survival function

When n=1, the survival function is a linearly decreasing function of s,

$$B(s) = 1 - \frac{s}{N} ,$$

$$b(s) = \frac{1}{N} \qquad 0\leq s\leq N.$$

Probabilistically interpreted, the life time has a uniform distribution on the interval [0,N]. Using the recurrence formula (70), we find

$$C_\rho^I(s;N,1) = \frac{1}{\rho}[1 - \frac{s}{N} - \frac{1}{N\rho}\{1-e^{-\rho(N-s)}\}],$$

and hence, from (73) and (74),

$$G_\rho(s) = \frac{(N-s)\rho-\{1-e^{-\rho(N-s)}\}}{N\rho - \{1-e^{-\rho N}\}} \xrightarrow[\rho\to 0]{} (1-\frac{s}{N})^2,$$

$$g_\rho(s) = \frac{1-e^{-\rho(N-s)}}{N-\{1-e^{-\rho N}\}/\rho} \xrightarrow[\rho\to 0]{} \frac{2}{N}(1-\frac{s}{N}) .$$

The vintage prices and the capital service price are in this case

$$q(t,s) = q(t)\,\frac{(N-s)\rho-\{1-e^{-\rho(N-s)}\}}{[N\rho-\{1-e^{-\rho N}\}](1-\frac{s}{N})} = q(t)\,\frac{N-s-H_\rho(N-s)}{[N-H_\rho(N)](1-\frac{s}{N})},$$

$$p(t,s) = q(t)\,\frac{(N-s)\rho-\{1-e^{-\rho(N-s)}\}}{N\rho-\{1-e^{-\rho N}\}} = q(t)\,\frac{N-s-H_\rho(N-s)}{N-H_\rho(N)},$$

$$c(t) = q(t)\,\frac{\rho}{1-\{1-e^{-\rho N}\}/(N\rho)} = q(t)\,\frac{\rho}{1-\frac{1}{N}H_\rho(N)} .$$

Finally, from (76) we find

$$\frac{E(S_N)}{E(S)} = \frac{2}{3} \qquad (\rho=0),$$

i.e. when the survival function is linearly decreasing, the expected life time of the net capital will be two thirds of the life time of the gross capital.

$n\geq 2$: Strictly convex survival functions

All members of this class of survival functions in which $n\geq 2$ are (strictly) convex functions of the age s. Or stated otherwise, the relative retirement (density) function b(s) is a decreasing function of s, since (67) implies

$$\frac{db(s)}{ds} = -\frac{n(n-1)}{N^2}(1-\frac{s}{N})^{n-2} < 0 \qquad \text{for } n\geq 2.$$

Moreover, b(s) is itself convex for $n>2$, since

$$\frac{d^2b(s)}{ds^2} = \frac{n(n-1)(n-2)}{N^3}(1-\frac{s}{N})^{n-3} > 0 \qquad \text{for } n>2.$$

This situation is illustrated in the upper half of figure 2.

In the limiting case where n goes to infinity while N is fixed, the survival function degenerates to

$$B(s) = B^I(s;N,\infty) = \begin{cases} 1 & \text{for } s=0 \\ 0 & \text{for } s>0, \end{cases}$$

i.e. the capital is scrapped momentaneously once it has been installed. On the other hand, if n and N *both* go to infinity while their ratio is restricted to be a finite constant δ, i.e.,

$n \to \infty$
$N \to \infty$
$\frac{n}{N} = \delta,$

we find from (66) and (67), when we recall the definition of e,

$$B(s) = (1 - \frac{s}{N})^n \to e^{-\delta s} ,$$

$$b(s) = \frac{n}{N}(1 - \frac{s}{N})^{n-1} \to \delta e^{-\delta s},$$

and hence, using (68) (or (70)),

$$C_\rho^I(s;N,n) \to \frac{e^{-\delta s}}{\rho+\delta} .$$

This limiting case is thus simply the exponential case discussed above.

CLASS II: Concave: $B(s) = 1 - (\frac{s}{N})^m$

The second class of survival functions we shall consider is

(80) $$B(s) = B^{II}(s;N,m) = \begin{cases} 1-(\frac{s}{N})^m & \text{for } 0 \leq s \leq N \\ 0 & \text{for } s>N, \end{cases}$$

where N and m are positive constants, m integer. Its retirement (density) function is

(81) $$b(s) = b^{II}(s;N,m) = \begin{cases} \frac{m}{N}(\frac{s}{N})^{m-1} & \text{for } 0 \leq s \leq N \\ 0 & \text{for } s>N. \end{cases}$$

For ease of exposition, we introduce the auxiliary function

(82) $$C_\rho^{II}(s;N,m) = \int_s^N e^{-\rho(z-s)}(\frac{z}{N})^m dz \qquad 0 \leq s \leq N.$$

When $\rho=0$, we find directly

(83) $$C_0^{II}(s;N,m) = \frac{N}{m+1}[1 - (\frac{s}{N})^{m+1}] = \frac{N}{m+1}B^{II}(s;N,m+1) \qquad 0 \leq s \leq N.$$

If $\rho>0$, the function values can be calculated recursively from the following formula, proved in appendix B:

(84) $$C_\rho^{II}(s;N,m) = \frac{1}{\rho}[(\frac{s}{N})^m - e^{-\rho(N-s)} + \frac{m}{N}C_\rho^{II}(s;N,m-1)]$$

$$= H_\rho(N-s) + \frac{1}{\rho}\{\frac{m}{N}C_\rho^{II}(s;N,m-1) - B^{II}(s;N,m)\} \qquad m=1,2,\ldots\ldots,$$

$$C_\rho^{II}(s;N,0) = H_\rho(N-s) = \frac{1}{\rho}[1-e^{-\rho(N-s)}].$$

Inserting (80) and (81) in (8) and (9), we find

$$(85)\qquad \Phi_\rho(s) = \frac{\frac{1}{\rho}[1-e^{-\rho(N-s)}] - C_\rho^{II}(s;N,m)}{B^{II}(s;N,m)} = \frac{H_\rho(N-s) - C_\rho^{II}(s;N,m)}{B^{II}(s;N,m)} \qquad 0\leq s\leq N,$$

$$(86)\qquad \psi_\rho(s) = \frac{m}{N}\frac{C_\rho^{II}(s;N,m-1)}{B^{II}(s;N,m)} \qquad 0\leq s\leq N.$$

We can then, by using (33) and (41), write the weighting functions for net capital and depreciation as follows

$$(87)\qquad G_\rho(s) = \frac{H_\rho(N-s) - C_\rho^{II}(s;N,m)}{H_\rho(N) - C_\rho^{II}(0;N,m)} ,$$

$$(88)\qquad g_\rho(s) = \frac{m}{N}\frac{C_\rho^{II}(s,N,m-1)}{H_\rho(N) - C_\rho^{II}(0;N,m)} .$$

For $\rho=0$ we get in particular

$$G_0(s) = 1 - \frac{s}{N} - \frac{s}{mN}[1 - (\frac{s}{N})^m] = (1 + \frac{1}{m})(1 - \frac{s}{N}) - \frac{1}{m}B^{II}(s;N,m+1) ,$$

$$g_0(s) = \frac{m+1}{mN}[1 - (\frac{s}{N})^m] = \frac{m+1}{mN} - \frac{1}{m}b^{II}(s;N,m+1).$$

Combining (23) with (80), (83), and (85), we get the following nonlinear expression for the expected remaining life time as a function of age:

$$(89)\qquad E(T;s) = \Phi_0(s) = \frac{N-s - \frac{N}{m+1}[1 - (\frac{s}{N})^{m+1}]}{1 - (\frac{s}{N})^m} = \frac{N}{m+1}\left[\frac{m(1 - \frac{s}{N})}{1 - (\frac{s}{N})^m} - \frac{s}{N}\right] .$$

In particular, we find by setting s=0 that the expected life time of a new capital unit is

$$(90)\qquad E(S) = \frac{mN}{m+1} .$$

The corresponding 'expected life time of net capital', as defined in (51), is

$$(91)\qquad E(S_N) = \frac{(m+1)N}{2(m+2)} ,$$

and thus their ratio,

$$(92)\qquad \frac{E(S_N)}{E(S)} = \frac{(m+1)^2}{2m(m+2)} = \frac{1}{2}[1 + \frac{1}{m(m+2)}],$$

decreases from 2/3 to 1/2 as m increases from 1 to infinity.

The expressions for the vintage prices are in this case

$$(93)\qquad q(t,s) = q(t)\,\frac{H_\rho(N-s) - C_\rho^{II}(s;N,m)}{\{H_\rho(N) - C_\rho^{II}(0;N,m)\}\{1 - (\frac{s}{N})^m\}}\,,$$

$$(94)\qquad p(t,s) = q(t)\,\frac{H_\rho(N-s) - C_\rho^{II}(s;N,m)}{H_\rho(N) - C_\rho^{II}(0;N,m)}\,,$$

and the capital service price is

$$(95)\qquad c(t) = \frac{q(t)}{H_\rho(N) - C_\rho^{II}(0;N,m)}\,.$$

This is also - as the corresponding formula in class I, (79)- a highly non-linear expression in ρ, N, and m. If we set $\rho=0$, while making use of (83), we can isolate the effect of the depreciation parameters N and m. This gives

$$q(t,s) = q(t)\left\{\frac{1-\frac{s}{N}}{1-(\frac{s}{N})^m} - \frac{s}{mN}\right\},$$

$$p(t,s) = q(t)\,\{1 - \frac{s}{N} - \frac{s}{mN}\,[1 - (\frac{s}{N})^m]\}\,,$$

$$c(t) = q(t)\,\frac{m+1}{mN}\,.$$

In this degenerate case, the capital service price is inversely proportional to the maximal life time N and proportional to (m+1)/m.

Let us consider briefly some special cases of this class of survival functions: First, when m=1, we are back again at the linear survival function. Second, if $m\to\infty$ (with N finite), the model degenerates to the simultaneous exit specification, since $\lim_{m\to\infty}(s/N)^m$ is zero when s<N, and one when s=N. These two cases were discussed above, as special cases of class I.

$m\geq 2$: Strictly concave survival functions

When $m\geq 2$, we see from (80) that the survival function is (strictly) concave in s, or stated otherwise, the relative retirement (density) function b(s) is an increasing function of s, since (81) implies

$$\frac{db(s)}{ds} = \frac{m(m-1)}{N^2}\,(\frac{s}{N})^{m-2} > 0 \qquad \text{for } m\geq 2.$$

An interesting property of this class of survival profiles is that the weighting functions of gross and net capital may have different curvature. If, for instance, $\rho=0$, the latter is convex, since

$$\frac{dG_0^2(s)}{ds^2} = -\frac{dg_0(s)}{ds} = \frac{m+1}{N^2}\,(\frac{s}{N})^{m-1} > 0,$$

while the former is, as already declared, concave. In this case, the retirement (density) function $b(s)$ and the corresponding depreciation function $g_0(s)$ also show different curvature; the former is convex,

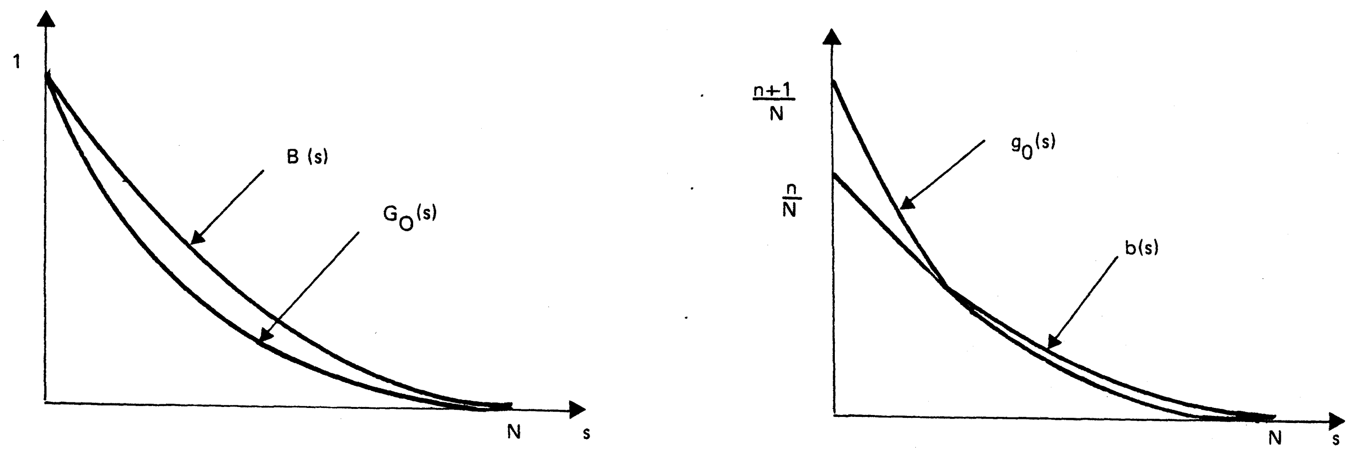
$$\frac{d^2b(s)}{ds^2} = \frac{m(m-1)(m-2)}{N^3} \left(\frac{s}{N}\right)^{m-3} \geq 0 \qquad \text{for } m \geq 2,$$

the latter concave,

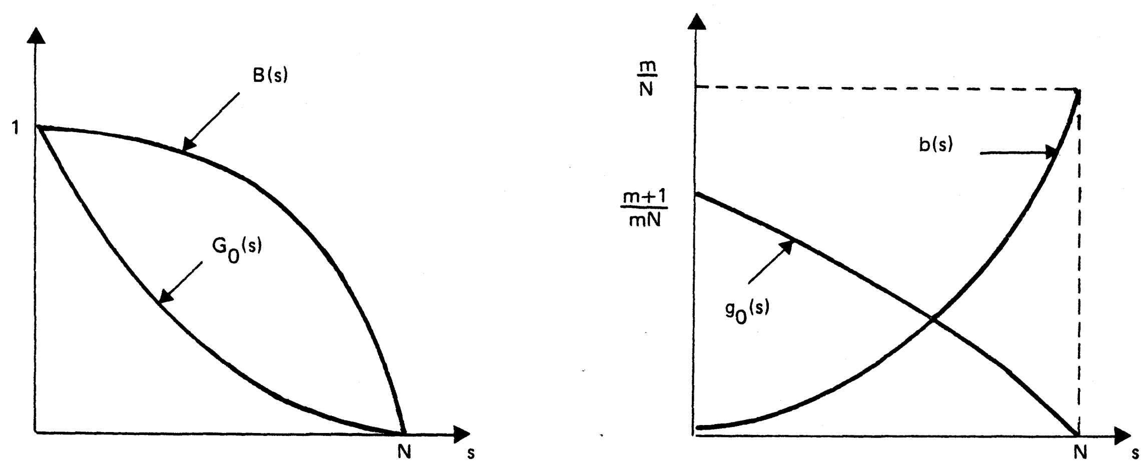
$$\frac{d^2g_0(s)}{ds^2} = - \frac{(m+1)(m-1)}{N^3} \left(\frac{s}{N}\right)^{m-2} < 0 \qquad \text{for } m \geq 2.$$

There is then no conflict between assuming that the *technical* outwear of the capital follows a *concave* function - i.e. that the deterioration is increasing with age - and assuming that the decline in the capital *value* is represented by a *convex* function - i.e. that the depreciation is decreasing with age. This situation illustrated in the lower half of figure 2.

FIGURE 2. TYPICAL CASES WITH CONVEX AND CONCAVE SURVIVAL PROFILES



B(s) and $G_0(s)$ both convex: Class I with $n \geq 2$.



B(s) concave, $G_0(s)$ convex: Class II with $m \geq 2$.

Numerical and graphical illustrations

Numerical illustrations of the functional forms in class I and II are given in tables 1-5. Function values of $B(s)$, $b(s)$, $G_\rho(s)$, $g_\rho(s)$, and $\Phi_\rho(s)$ are calculated from the formulae above[18] for a maximal life time N equal to 6, 20, and 50, and an interest rate ρ equal to 0 and 0.10. The different values of the 'curvature parameters' n and m considered illustrate the flexibility of these two parametrizations.

More detailed, graphical illustrations for N=20 are given in figures 3-13. Figures 3 and 4 contain the survival function and the corresponding replacement (density) function in class I for n=1, 2, 5, and 10. Figures 5 and 6 give similar functions for class II. Figures 7-9 visualize the difference between the survival function and the corresponding weighting function for net capital, whereas figures 10 and 11 illustrate these differences in terms of the replacement (density) function and the depreciation function. The function $\Phi_\rho(s)$, which represents the discounted service flow per capital (efficiency) unit as a function of age, is illustrated in figures 12 and 13. Recall that the graphs for $\rho=0$ also indicate the decline in the expected remaining life time.

Finally, in figures 14 and 15, we illustrate the decline in the vintage prices $q(t,s)$ as a proportion of the price of a corresponding new capital unit, $q(t)$. Since this ratio is equal to $\Phi_\rho(s)/\Phi_\rho(0)$, cf. (27), the graphs in figures 14 and 15 simply emerge by rescaling the graphs in figures 12 and 13.

18) In these calculations we did not, however, use the recurrence formulae (70) and (84) for the computation of the functions $C_\rho^I(s;N,n)$ and $C_\rho^{II}(s;N,m)$, since this algorithm turned out to give numerically imprecise results owing to cumulative rounding errors, in particular when ρ is small and n, or m, is large. Instead, we programmed the computer algorithm directly from the definitions (68) and (82), using Simpson's formula, which turned out to give more accurate results. In terms of computer time, these two procedures are largely equivalent.

TABLE 1. Survival functions for gross capital , B(s).

Class I and II.

Technical life time: N = 6, 20, and 50.

N = 6

s	simult. exit	CLASS II			CLASS I			
		m=10	m=5	m=2	n=1	n=2	n=5	n=10
00	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
01	1.00000	1.00000	0.99987	0.97222	0.83333	0.69444	0.40188	0.16151
02	1.00000	0.99998	0.99588	0.88889	0.66667	0.44444	0.13169	0.01734
03	1.00000	0.99902	0.96875	0.75000	0.50000	0.25000	0.03125	0.00098
04	1.00000	0.98266	0.86831	0.55556	0.33333	0.11111	0.00412	0.00002
05	1.00000	0.83849	0.59812	0.30556	0.16667	0.02778	0.00013	0.00000
06	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

TABLE 1 (cont.)

N = 20

s	simult. exit	CLASS II			CLASS I			
		m=10	m=5	m=2	n=1	n=2	n=5	n=10
00	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
01	1.00000	1.00000	1.00000	0.99750	0.95000	0.90250	0.77378	0.59874
02	1.00000	1.00000	0.99999	0.99000	0.90000	0.81000	0.59049	0.34868
03	1.00000	1.00000	0.99992	0.97750	0.85000	0.72250	0.44371	0.19687
04	1.00000	1.00000	0.99968	0.96000	0.80000	0.64000	0.32768	0.10737
05	1.00000	1.00000	0.99902	0.93750	0.75000	0.56250	0.23730	0.05631
06	1.00000	0.99999	0.99757	0.91000	0.70000	0.49000	0.16807	0.02825
07	1.00000	0.99997	0.99475	0.87750	0.65000	0.42250	0.11603	0.01346
08	1.00000	0.99990	0.98976	0.84000	0.60000	0.36000	0.07776	0.00605
09	1.00000	0.99966	0.98155	0.79750	0.55000	0.30250	0.05033	0.00253
10	1.00000	0.99902	0.96875	0.75000	0.50000	0.25000	0.03125	0.00098
11	1.00000	0.99747	0.94967	0.69750	0.45000	0.20250	0.01845	0.00034
12	1.00000	0.99395	0.92224	0.64000	0.40000	0.16000	0.01024	0.00010
13	1.00000	0.98654	0.88397	0.57750	0.35000	0.12250	0.00525	0.00003
14	1.00000	0.97175	0.83193	0.51000	0.30000	0.09000	0.00243	0.00001
15	1.00000	0.94369	0.76270	0.43750	0.25000	0.06250	0.00098	0.00000
16	1.00000	0.89263	0.67232	0.36000	0.20000	0.04000	0.00032	0.00000
17	1.00000	0.80313	0.55629	0.27750	0.15000	0.02250	0.00008	0.00000
18	1.00000	0.65132	0.40951	0.19000	0.10000	0.01000	0.00001	0.00000
19	1.00000	0.40126	0.22622	0.09750	0.05000	0.00250	0.00000	0.00000
20	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

TABLE 1 (cont.)

N = 50

s	simult. exit	CLASS II m=10	CLASS II m=5	CLASS II n=2	CLASS I n=1	CLASS I n=2	CLASS I n=5	CLASS I n=10
00	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
01	1.00000	1.00000	1.00000	0.99960	0.98000	0.96040	0.90392	0.81707
02	1.00000	1.00000	1.00000	0.99840	0.96000	0.92160	0.81537	0.66483
03	1.00000	1.00000	1.00000	0.99640	0.94000	0.88360	0.73390	0.53862
04	1.00000	1.00000	1.00000	0.99360	0.92000	0.84640	0.65908	0.43439
05	1.00000	1.00000	0.99999	0.99000	0.90000	0.81000	0.59049	0.34868
06	1.00000	1.00000	0.99998	0.98560	0.88000	0.77440	0.52773	0.27850
07	1.00000	1.00000	0.99995	0.98040	0.86000	0.73960	0.47043	0.22130
08	1.00000	1.00000	0.99990	0.97440	0.84000	0.70560	0.41821	0.17490
09	1.00000	1.00000	0.99981	0.96760	0.82000	0.67240	0.37074	0.13745
10	1.00000	1.00000	0.99968	0.96000	0.80000	0.64000	0.32768	0.10737
11	1.00000	1.00000	0.99948	0.95160	0.78000	0.60840	0.28872	0.08336
12	1.00000	1.00000	0.99920	0.94240	0.76000	0.57760	0.25355	0.06429
13	1.00000	1.00000	0.99881	0.93240	0.74000	0.54760	0.22190	0.04924
14	1.00000	1.00000	0.99828	0.92160	0.72000	0.51840	0.19349	0.03744
15	1.00000	0.99999	0.99757	0.91000	0.70000	0.49000	0.16807	0.02825
16	1.00000	0.99999	0.99664	0.89760	0.68000	0.46240	0.14539	0.02114
17	1.00000	0.99998	0.99546	0.88440	0.66000	0.43560	0.12523	0.01568
18	1.00000	0.99996	0.99395	0.87040	0.64000	0.40960	0.10737	0.01153
19	1.00000	0.99994	0.99208	0.85560	0.62000	0.38440	0.09161	0.00839
20	1.00000	0.99990	0.98976	0.84000	0.60000	0.36000	0.07776	0.00605
21	1.00000	0.99983	0.98693	0.82360	0.58000	0.33640	0.06564	0.00431
22	1.00000	0.99973	0.98351	0.80640	0.56000	0.31360	0.05507	0.00303
23	1.00000	0.99958	0.97940	0.78840	0.54000	0.29160	0.04592	0.00211

TABLE 1 (cont.)

N = 50 (cont.)

s	simult. exit	CLASS II			CLASS I			
		m=10	m=5	m=2	n=1	n=2	n=5	n=10
24	1.00000	0.99935	0.97452	0.76960	0.52000	0.27040	0.03802	0.00145
25	1.00000	0.99902	0.96875	0.75000	0.50000	0.25000	0.03125	0.00098
26	1.00000	0.99855	0.96198	0.72960	0.48000	0.23040	0.02548	0.00065
27	1.00000	0.99789	0.95408	0.70840	0.46000	0.21160	0.02060	0.00042
28	1.00000	0.99697	0.94493	0.68640	0.44000	0.19360	0.01649	0.00027
29	1.00000	0.99569	0.93436	0.66360	0.42000	0.17640	0.01307	0.00017
30	1.00000	0.99395	0.92224	0.64000	0.40000	0.16000	0.01024	0.00010
31	1.00000	0.99161	0.90839	0.61560	0.38000	0.14440	0.00792	0.00006
32	1.00000	0.98847	0.89263	0.59040	0.36000	0.12960	0.00605	0.00004
33	1.00000	0.98432	0.87477	0.56440	0.34000	0.11560	0.00454	0.00002
34	1.00000	0.97886	0.85461	0.53760	0.32000	0.10240	0.00336	0.00001
35	1.00000	0.97175	0.83193	0.51000	0.30000	0.09000	0.00243	0.00001
36	1.00000	0.96256	0.80651	0.48160	0.28000	0.07840	0.00172	0.00000
37	1.00000	0.95076	0.77810	0.45240	0.26000	0.06760	0.00119	0.00000
38	1.00000	0.93571	0.74645	0.42240	0.24000	0.05760	0.00080	0.00000
39	1.00000	0.91654	0.71128	0.39160	0.22000	0.04840	0.00052	0.00000
40	1.00000	0.89263	0.67232	0.36000	0.20000	0.04000	0.00032	0.00000
41	1.00000	0.86255	0.62926	0.32760	0.18000	0.03240	0.00019	0.00000
42	1.00000	0.82510	0.58179	0.29440	0.16000	0.02560	0.00010	0.00000
43	1.00000	0.77670	0.52957	0.26040	0.14000	0.01960	0.00005	0.00000
44	1.00000	0.72150	0.47227	0.22560	0.12000	0.01440	0.00002	0.00000
45	1.00000	0.65132	0.40951	0.19000	0.10000	0.01000	0.00001	0.00000
46	1.00000	0.56561	0.34092	0.15360	0.08000	0.00640	0.00000	0.00000
47	1.00000	0.46139	0.26610	0.11640	0.06000	0.00360	0.00000	0.00000
48	1.00000	0.33517	0.18463	0.07840	0.04000	0.00160	0.00000	0.00000
49	1.00000	0.18293	0.09608	0.03960	0.02000	0.00040	0.00000	0.00000
50	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

TABLE 2. Replacement (density) functions, b(s).
Class I and II.
Technical life time: N = 6, 20, and 50.

N = 6

s	simult. exit	CLASS II			CLASS I			
		m=10	m=5	m=2	n=1	n=2	n=5	n=10
00	0.00000	0.00000	0.00000	0.00000	0.16667	0.33333	0.83333	1.66667
01	0.00000	0.00000	0.00064	0.05556	0.16667	0.27778	0.40188	0.32301
02	0.00000	0.00008	0.01029	0.11111	0.16667	0.22222	0.16461	0.04335
03	0.00000	0.00326	0.05208	0.16667	0.16667	0.16667	0.05208	0.00326
04	0.00000	0.04335	0.16461	0.22222	0.16667	0.11111	0.01029	0.00008
05	0.00000	0.32301	0.40188	0.27778	0.16667	0.05556	0.00064	0.00000
06	NA	1.66667	0.83333	0.33333	0.16667	0.00000	0.00000	0.00000

TABLE 2 (cont.)

N = 20

s	simult. exit	CLASS II m=10	CLASS II m=5	CLASS II m=2	CLASS I n=1	CLASS I n=2	CLASS I n=5	CLASS I n=10
00	0.00000	0.00000	0.00000	0.00000	0.05000	0.10000	0.25000	0.50000
01	0.00000	0.00000	0.00000	0.00500	0.05000	0.09500	0.20363	0.31512
02	0.00000	0.00000	0.00002	0.01000	0.05000	0.09000	0.16403	0.19371
03	0.00000	0.00000	0.00013	0.01500	0.05000	0.08500	0.13050	0.11581
04	0.00000	0.00000	0.00040	0.02000	0.05000	0.08000	0.10240	0.06711
05	0.00000	0.00000	0.00098	0.02500	0.05000	0.07500	0.07910	0.03754
06	0.00000	0.00001	0.00202	0.03000	0.05000	0.07000	0.06002	0.02018
07	0.00000	0.00004	0.00375	0.03500	0.05000	0.06500	0.04463	0.01036
08	0.00000	0.00013	0.00640	0.04000	0.05000	0.06000	0.03240	0.00504
09	0.00000	0.00038	0.01025	0.04500	0.05000	0.05500	0.02288	0.00230
10	0.00000	0.00098	0.01563	0.05000	0.05000	0.05000	0.01563	0.00098
11	0.00000	0.00230	0.02288	0.05500	0.05000	0.04500	0.01025	0.00038
12	0.00000	0.00504	0.03240	0.06000	0.05000	0.04000	0.00640	0.00013
13	0.00000	0.01036	0.04463	0.06500	0.05000	0.03500	0.00375	0.00004
14	0.00000	0.02018	0.06002	0.07000	0.05000	0.03000	0.00202	0.00001
15	0.00000	0.03754	0.07910	0.07500	0.05000	0.02500	0.00098	0.00000
16	0.00000	0.06711	0.10240	0.08000	0.05000	0.02000	0.00040	0.00000
17	0.00000	0.11581	0.13050	0.08500	0.05000	0.01500	0.00013	0.00000
18	0.00000	0.19371	0.16402	0.09000	0.05000	0.01000	0.00003	0.00000
19	0.00000	0.31512	0.20363	0.09500	0.05000	0.00500	0.00000	0.00000
20	NA	0.50000	0.25000	0.10000	0.05000	0.00000	0.00000	0.00000

TABLE 2 (cont.)

N = 50

s	simult. exit	CLASS II			CLASS I			
		m=10	m=5	m=2	n=1	n=2	n=5	n=10
00	0.00000	0.00000	0.00000	0.00000	0.02000	0.04000	0.10000	0.20000
01	0.00000	0.00000	0.00000	0.00080	0.02000	0.03920	0.09224	0.16675
02	0.00000	0.00000	0.00000	0.00160	0.02000	0.03840	0.08493	0.13851
03	0.00000	0.00000	0.00000	0.00240	0.02000	0.03760	0.07807	0.11460
04	0.00000	0.00000	0.00000	0.00320	0.02000	0.03680	0.07164	0.09443
05	0.00000	0.00000	0.00001	0.00400	0.02000	0.03600	0.06561	0.07748
06	0.00000	0.00000	0.00002	0.00480	0.02000	0.03520	0.05997	0.06330
07	0.00000	0.00000	0.00004	0.00560	0.02000	0.03440	0.05470	0.05147
08	0.00000	0.00000	0.00007	0.00640	0.02000	0.03360	0.04979	0.04164
09	0.00000	0.00000	0.00010	0.00720	0.02000	0.03280	0.04521	0.03352
10	0.00000	0.00000	0.00016	0.00800	0.02000	0.03200	0.04096	0.02684
11	0.00000	0.00000	0.00023	0.00880	0.02000	0.03120	0.03702	0.02137
12	0.00000	0.00000	0.00033	0.00960	0.02000	0.03040	0.03336	0.01692
13	0.00000	0.00000	0.00046	0.01040	0.02000	0.02960	0.02999	0.01331
14	0.00000	0.00000	0.00061	0.01120	0.02000	0.02880	0.02687	0.01040
15	0.00000	0.00000	0.00081	0.01200	0.02000	0.02800	0.02401	0.00807
16	0.00000	0.00001	0.00105	0.01280	0.02000	0.02720	0.02138	0.00622
17	0.00000	0.00001	0.00134	0.01360	0.02000	0.02640	0.01897	0.00475
18	0.00000	0.00002	0.00168	0.01440	0.02000	0.02560	0.01678	0.00360
19	0.00000	0.00003	0.00209	0.01520	0.02000	0.02480	0.01478	0.00271
20	0.00000	0.00005	0.00256	0.01600	0.02000	0.02400	0.01296	0.00202
21	0.00000	0.00008	0.00311	0.01680	0.02000	0.02320	0.01132	0.00149
22	0.00000	0.00012	0.00375	0.01760	0.02000	0.02240	0.00983	0.00108
23	0.00000	0.00018	0.00448	0.01840	0.02000	0.02160	0.00850	0.00078

TABLE 2 (cont.)

N = 50 (cont.)

s	simult. exit	CLASS II			CLASS I			
		m=10	m=5	m=2	n=1	n=2	n=5	n=10
24	0.00000	0.00027	0.00531	0.01920	0.02000	0.02080	0.00731	0.00056
25	0.00000	0.00039	0.00625	0.02000	0.02000	0.02000	0.00625	0.00039
26	0.00000	0.00056	0.00731	0.02080	0.02000	0.01920	0.00531	0.00027
27	0.00000	0.00078	0.00850	0.02160	0.02000	0.01840	0.00448	0.00018
28	0.00000	0.00108	0.00983	0.02240	0.02000	0.01760	0.00375	0.00012
29	0.00000	0.00149	0.01132	0.02320	0.02000	0.01680	0.00311	0.00008
30	0.00000	0.00202	0.01296	0.02400	0.02000	0.01600	0.00256	0.00005
31	0.00000	0.00271	0.01478	0.02480	0.02000	0.01520	0.00209	0.00003
32	0.00000	0.00360	0.01678	0.02560	0.02000	0.01440	0.00168	0.00002
33	0.00000	0.00475	0.01897	0.02640	0.02000	0.01360	0.00134	0.00001
34	0.00000	0.00622	0.02138	0.02720	0.02000	0.01280	0.00105	0.00001
35	0.00000	0.00807	0.02401	0.02800	0.02000	0.01200	0.00081	0.00000
36	0.00000	0.01040	0.02687	0.02880	0.02000	0.01120	0.00061	0.00000
37	0.00000	0.01331	0.02999	0.02960	0.02000	0.01040	0.00046	0.00000
38	0.00000	0.01692	0.03336	0.03040	0.02000	0.00960	0.00033	0.00000
39	0.00000	0.02137	0.03702	0.03120	0.02000	0.00880	0.00023	0.00000
40	0.00000	0.02684	0.04096	0.03200	0.02000	0.00800	0.00016	0.00000
41	0.00000	0.03352	0.04521	0.03280	0.02000	0.00720	0.00010	0.00000
42	0.00000	0.04164	0.04979	0.03360	0.02000	0.00640	0.00007	0.00000
43	0.00000	0.05147	0.05470	0.03440	0.02000	0.00560	0.00004	0.00000
44	0.00000	0.06330	0.05997	0.03520	0.02000	0.00480	0.00002	0.00000
45	0.00000	0.07748	0.06561	0.03600	0.02000	0.00400	0.00001	0.00000
46	0.00000	0.09443	0.07164	0.03680	0.02000	0.00320	0.00000	0.00000
47	0.00000	0.11450	0.07807	0.03760	0.02000	0.00240	0.00000	0.00000
48	0.00000	0.13851	0.08493	0.03840	0.02000	0.00160	0.00000	0.00000
49	0.00000	0.16675	0.09224	0.03920	0.02000	0.00080	0.00000	0.00000
50	NA	0.20000	0.10000	0.04000	0.02000	0.00000	0.00000	0.00000

TABLE 3. Weighting functions for net capital, G(s).

Class I and II.

Technical life time: N = 6, 20, and 50.

Interest rate : Rho (ρ) = 0 and 0.10.

N = 6

s	Rho = 0				Rho = 0.10			
	simult. exit	class II m=5	class I n=1	class I n=5	simult. exit	class II m=5	class I n=1	class I n=5
00	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
01	0.83333	0.80000	0.69444	0.33490	0.87207	0.83635	0.71588	0.33944
02	0.66667	0.60027	0.44444	0.08779	0.73069	0.65584	0.47254	0.09020
03	0.50000	0.40312	0.25000	0.01563	0.57444	0.45977	0.27429	0.01628
04	0.33333	0.21756	0.11111	0.00137	0.40176	0.25848	0.12587	0.00145
05	0.16667	0.06698	0.02778	0.00002	0.21092	0.08266	0.03251	0.00002
06	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

TABLE 3 (cont.)

N = 20

s	Rho = 0				Rho = 0.10			
	simult. exit	class II m=5	class I n=1	class I n=5	simult. exit	class II m=5	class I n=1	class I n=5
00	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
01	0.95000	0.94000	0.90250	0.73509	0.98354	0.97412	0.92446	0.74376
02	0.90000	0.88000	0.81000	0.53144	0.96535	0.94553	0.85023	0.54410
03	0.85000	0.82000	0.72250	0.37715	0.94524	0.91393	0.77746	0.39078
04	0.80000	0.76001	0.64000	0.26214	0.92302	0.87903	0.70631	0.27491
05	0.75000	0.70005	0.56250	0.17798	0.89846	0.84051	0.63693	0.18893
06	0.70000	0.64015	0.49000	0.11765	0.87132	0.79807	0.56952	0.12644
07	0.65000	0.58037	0.42250	0.07542	0.84133	0.75144	0.50428	0.08207
08	0.60000	0.52082	0.36000	0.04666	0.80818	0.70040	0.44145	0.05141
09	0.55000	0.46166	0.30250	0.02768	0.77155	0.64485	0.38127	0.03089
10	0.50000	0.40313	0.25000	0.01562	0.73106	0.58480	0.32403	0.01766
11	0.45000	0.34554	0.20250	0.00830	0.68631	0.52051	0.27003	0.00951
12	0.40000	0.28933	0.16000	0.00410	0.63686	0.45246	0.21961	0.00475
13	0.35000	0.23508	0.12250	0.00184	0.58221	0.38153	0.17315	0.00216
14	0.30000	0.18353	0.09000	0.00073	0.52181	0.30900	0.13107	0.00087
15	0.25000	0.13560	0.06250	0.00024	0.45505	0.23673	0.09383	0.00029
16	0.20000	0.09243	0.04000	0.00006	0.38128	0.16724	0.06194	0.00008
17	0.15000	0.05543	0.02250	0.00001	0.29975	0.10390	0.03595	0.00001
18	0.10000	0.02629	0.01000	0.00000	0.20964	0.05102	0.01650	0.00000
19	0.05000	0.00702	0.00250	0.00000	0.11006	0.01409	0.00426	0.00000
20	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

TABLE 3 (cont.)

N = 50

s	Rho = 0 simult. exit	Rho = 0 class II m=5	Rho = 0 class I n=1	Rho = 0 class I n=5	Rho = 0.10 simult. exit	Rho = 0.10 class II m=5	Rho = 0.10 class I n=1	Rho = 0.10 class I n=5
00	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
01	0.98000	0.97600	0.96040	0.88584	0.99929	0.99769	0.97522	0.89385
02	0.96000	0.95200	0.92160	0.78276	0.99850	0.99514	0.95046	0.79703
03	0.94000	0.92800	0.88360	0.68987	0.99763	0.99232	0.92571	0.70891
04	0.92000	0.90400	0.84640	0.60635	0.99666	0.98920	0.90099	0.62886
05	0.90000	0.88000	0.81000	0.53144	0.99560	0.98576	0.87630	0.55631
06	0.88000	0.85600	0.77440	0.46440	0.99442	0.98195	0.85163	0.49072
07	0.86000	0.83200	0.73960	0.40457	0.99312	0.97775	0.82700	0.43155
08	0.84000	0.80800	0.70560	0.35130	0.99169	0.97311	0.80239	0.37832
09	0.82000	0.78401	0.67240	0.30401	0.99010	0.96798	0.77783	0.33055
10	0.80000	0.76001	0.64000	0.26214	0.98834	0.96234	0.75331	0.28781
11	0.78000	0.73602	0.60840	0.22520	0.98640	0.95611	0.72883	0.24968
12	0.76000	0.71204	0.57760	0.19270	0.98426	0.94925	0.70440	0.21576
13	0.74000	0.68806	0.54760	0.16421	0.98189	0.94171	0.68003	0.18569
14	0.72000	0.66410	0.51840	0.13931	0.97927	0.93343	0.65572	0.15913
15	0.70000	0.64015	0.49000	0.11765	0.97638	0.92434	0.63148	0.13575
16	0.68000	0.61621	0.46240	0.09887	0.97318	0.91438	0.60732	0.11525
17	0.66000	0.59231	0.43560	0.08265	0.96965	0.90349	0.58324	0.09735
18	0.64000	0.56844	0.40960	0.06872	0.96574	0.89159	0.55925	0.08178
19	0.62000	0.54460	0.38440	0.05680	0.96143	0.87862	0.53536	0.06831
20	0.60000	0.52082	0.36000	0.04666	0.95666	0.86452	0.51158	0.05671
21	0.58000	0.49710	0.33640	0.03807	0.95139	0.84921	0.48793	0.04677
22	0.56000	0.47345	0.31360	0.03084	0.94556	0.83262	0.46442	0.03830
23	0.54000	0.44989	0.29160	0.02479	0.93912	0.81469	0.44106	0.03113

TABLE 3 (cont.)

N = 50 (cont.)

s	Rho = 0				Rho = 0.10			
	simult. exit	class II m=5	class I n=1	class I n=5	simult. exit	class II m=5	class I n=1	class I n=5
24	0.52000	0.42645	0.27040	0.01977	0.93201	0.79535	0.41786	0.02509
25	0.50000	0.40312	0.25000	0.01562	0.92414	0.77456	0.39485	0.02005
26	0.48000	0.37995	0.23040	0.01223	0.91545	0.75225	0.37205	0.01587
27	0.46000	0.35696	0.21160	0.00947	0.90584	0.72837	0.34947	0.01243
28	0.44000	0.33417	0.19360	0.00726	0.89523	0.70290	0.32715	0.00963
29	0.42000	0.31161	0.17640	0.00549	0.88350	0.67581	0.30510	0.00737
30	0.40000	0.28933	0.16000	0.00410	0.87053	0.64709	0.28335	0.00556
31	0.38000	0.26736	0.14440	0.00301	0.85620	0.61674	0.26195	0.00414
32	0.36000	0.24574	0.12960	0.00218	0.84036	0.58478	0.24092	0.00303
33	0.34000	0.22453	0.11560	0.00154	0.82286	0.55127	0.22030	0.00217
34	0.32000	0.20377	0.10240	0.00107	0.80352	0.51627	0.20014	0.00153
35	0.30000	0.18353	0.09000	0.00073	0.78214	0.47989	0.18048	0.00105
36	0.28000	0.16386	0.07840	0.00048	0.75851	0.44226	0.16138	0.00070
37	0.26000	0.14484	0.06760	0.00031	0.73240	0.40356	0.14289	0.00046
38	0.24000	0.12654	0.05760	0.00019	0.70355	0.36402	0.12509	0.00029
39	0.22000	0.10904	0.04840	0.00011	0.67165	0.32390	0.10803	0.00017
40	0.20000	0.09243	0.04000	0.00006	0.63641	0.28354	0.09181	0.00010
41	0.18000	0.07680	0.03240	0.00003	0.59746	0.24333	0.07651	0.00005
42	0.16000	0.06226	0.02560	0.00002	0.55441	0.20376	0.06223	0.00003
43	0.14000	0.04891	0.01960	0.00001	0.50683	0.16537	0.04906	0.00001
44	0.12000	0.03688	0.01440	0.00000	0.45425	0.12883	0.03714	0.00000
45	0.10000	0.02629	0.01000	0.00000	0.39614	0.09488	0.02659	0.00000
46	0.08000	0.01727	0.00640	0.00000	0.33192	0.06442	0.01755	0.00000
47	0.06000	0.00997	0.00360	0.00000	0.26094	0.03845	0.01019	0.00000
48	0.04000	0.00455	0.00160	0.00000	0.18250	0.01814	0.00467	0.00000
49	0.02000	0.00117	0.00040	0.00000	0.09581	0.00491	0.00121	0.00000
50	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

TABLE 4. Depreciation (density) functions, g(s).

Class I and II.

Technical life time: N =6, 20, and 50.

Interest rate: Rho (ρ) = 0 and 0.10.

N = 6

s	Rho = 0				Rho = 0.10			
	simult. exit	class II m=5	class I n=1	class I n=5	simult. exit	class II m=5	class I n=1	class I n=5
00	0.16667	0.20000	0.33333	1.00000	0.12164	0.15561	0.30320	0.98659
01	0.16667	0.19997	0.27778	0.40188	0.13443	0.17194	0.26441	0.40273
02	0.16667	0.19918	0.22222	0.13169	0.14857	0.18897	0.22154	0.13407
03	0.16667	0.19375	0.16667	0.03125	0.16419	0.20164	0.17417	0.03233
04	0.16667	0.17366	0.11111	0.00412	0.18146	0.19610	0.12181	0.00433
05	0.16667	0.11962	0.05556	0.00013	0.20055	0.14462	0.06395	0.00014
06	0.16667	0.00000	0.00000	0.00000	0.22164	0.00000	0.00000	0.00000

TABLE 4 (cont.)

N = 20

s	Rho = 0 simult. exit	Rho = 0 class II m=5	Rho = 0 class I n=1	Rho = 0 class I n=5	Rho = 0.10 simult. exit	Rho = 0.10 class II m=5	Rho = 0.10 class I n=1	Rho = 0.10 class I n=5
.00	0.05000	0.06000	0.10000	0.30000	0.01565	0.02460	0.07616	0.28833
.01	0.05000	0.06000	0.09500	0.23213	0.01730	0.02719	0.07491	0.22611
.02	0.05000	0.06000	0.09000	0.17715	0.01912	0.03005	0.07352	0.17490
.03	0.05000	0.06000	0.08500	0.13311	0.02113	0.03320	0.07199	0.13323
.04	0.05000	0.05998	0.08000	0.09830	0.02335	0.03666	0.07030	0.09976
.05	0.05000	0.05994	0.07500	0.07119	0.02581	0.04043	0.06843	0.07326
.06	0.05000	0.05985	0.07000	0.05042	0.02852	0.04449	0.06636	0.05262
.07	0.05000	0.05968	0.06500	0.03481	0.03152	0.04880	0.06408	0.03685
.08	0.05000	0.05939	0.06000	0.02333	0.03483	0.05329	0.06155	0.02506
.09	0.05000	0.05889	0.05500	0.01510	0.03850	0.05782	0.05876	0.01646
.10	0.05000	0.05813	0.05000	0.00937	0.04255	0.06223	0.05568	0.01037
.11	0.05000	0.05698	0.04500	0.00554	0.04702	0.06628	0.05227	0.00622
.12	0.05000	0.05533	0.04000	0.00307	0.05197	0.06967	0.04850	0.00350
.13	0.05000	0.05304	0.03500	0.00158	0.05743	0.07199	0.04434	0.00182
.14	0.05000	0.04992	0.03000	0.00073	0.06347	0.07276	0.03974	0.00086
.15	0.05000	0.04576	0.02500	0.00029	0.07015	0.07136	0.03466	0.00035
.16	0.05000	0.04034	0.02000	0.00010	0.07752	0.06705	0.02904	0.00012
.17	0.05000	0.03338	0.01500	0.00002	0.08568	0.05893	0.02283	0.00003
.18	0.05000	0.02457	0.01000	0.00000	0.09469	0.04592	0.01597	0.00000
.19	0.05000	0.01357	0.00500	0.00000	0.10465	0.02678	0.00838	0.00000
.20	0.05000	0.00000	0.00000	0.00000	0.11565	0.00000	0.00000	0.00000

TABLE 4 (cont.)

N = 50

s	Rho = 0				Rho = 0.10			
	simult. exit	class II m=5	class I n=1	class I n=5	simult. exit	class II m=5	class I n=1	class I n=5
00	0.02000	0.02400	0.04000	0.12000	0.00068	0.00220	0.02479	0.11102
01	0.02000	0.02400	0.03920	0.10847	0.00075	0.00243	0.02477	0.10136
02	0.02000	0.02400	0.03840	0.09784	0.00083	0.00268	0.02475	0.09236
03	0.02000	0.02400	0.03760	0.08807	0.00092	0.00296	0.02473	0.08398
04	0.02000	0.02400	0.03680	0.07909	0.00101	0.00328	0.02471	0.07620
05	0.02000	0.02400	0.03600	0.07086	0.00112	0.00362	0.02468	0.06898
06	0.02000	0.02400	0.03520	0.06333	0.00124	0.00400	0.02465	0.06229
07	0.02000	0.02400	0.03440	0.05645	0.00137	0.00442	0.02462	0.05612
08	0.02000	0.02400	0.03360	0.05019	0.00151	0.00487	0.02458	0.05042
09	0.02000	0.02400	0.03280	0.04449	0.00167	0.00538	0.02454	0.04518
10	0.02000	0.02399	0.03200	0.03932	0.00184	0.00593	0.02450	0.04037
11	0.02000	0.02399	0.03120	0.03465	0.00204	0.00653	0.02445	0.03596
12	0.02000	0.02398	0.03040	0.03043	0.00225	0.00719	0.02440	0.03193
13	0.02000	0.02397	0.02960	0.02663	0.00249	0.00790	0.02434	0.02826
14	0.02000	0.02396	0.02880	0.02322	0.00275	0.00868	0.02428	0.02492
15	0.02000	0.02394	0.02800	0.02017	0.00304	0.00951	0.02420	0.02189
16	0.02000	0.02392	0.02720	0.01745	0.00336	0.01041	0.02412	0.01916
17	0.02000	0.02389	0.02640	0.01503	0.00371	0.01138	0.02404	0.01669
18	0.02000	0.02385	0.02560	0.01288	0.00410	0.01242	0.02394	0.01448
19	0.02000	0.02381	0.02480	0.01099	0.00454	0.01352	0.02383	0.01250
20	0.02000	0.02375	0.02400	0.00933	0.00501	0.01470	0.02372	0.01074
21	0.02000	0.02369	0.02320	0.00788	0.00554	0.01594	0.02358	0.00917
22	0.02000	0.02360	0.02240	0.00661	0.00612	0.01725	0.02344	0.00779
23	0.02000	0.02351	0.02160	0.00551	0.00677	0.01862	0.02328	0.00658

TABLE 4 (cont.)

N = 50 (cont.)

s	Rho = 0				Rho = 0.10			
	simult. exit	class II m=5	class I n=1	class I n=5	simult. exit	class II m=5	class I n=1	class I n=5
24	0.02000	0.02339	0.02080	0.00456	0.00748	0.02006	0.02310	0.00551
25	0.02000	0.02325	0.02000	0.00375	0.00826	0.02155	0.02291	0.00459
26	0.02000	0.02309	0.01920	0.00306	0.00913	0.02309	0.02269	0.00379
27	0.02000	0.02290	0.01840	0.00247	0.01009	0.02467	0.02246	0.00310
28	0.02000	0.02268	0.01760	0.00198	0.01116	0.02628	0.02219	0.00252
29	0.02000	0.02242	0.01680	0.00157	0.01233	0.02791	0.02190	0.00202
30	0.02000	0.02213	0.01600	0.00123	0.01363	0.02954	0.02158	0.00160
31	0.02000	0.02180	0.01520	0.00095	0.01506	0.03116	0.02122	0.00126
32	0.02000	0.02142	0.01440	0.00073	0.01664	0.03274	0.02083	0.00097
33	0.02000	0.02099	0.01360	0.00055	0.01839	0.03427	0.02040	0.00074
34	0.02000	0.02051	0.01280	0.00040	0.02033	0.03571	0.01992	0.00056
35	0.02000	0.01997	0.01200	0.00029	0.02246	0.03703	0.01939	0.00041
36	0.02000	0.01936	0.01120	0.00021	0.02483	0.03820	0.01880	0.00029
37	0.02000	0.01867	0.01040	0.00014	0.02744	0.03916	0.01816	0.00021
38	0.02000	0.01791	0.00960	0.00010	0.03032	0.03988	0.01744	0.00014
39	0.02000	0.01707	0.00880	0.00006	0.03351	0.04030	0.01665	0.00009
40	0.02000	0.01614	0.00800	0.00004	0.03704	0.04035	0.01578	0.00005
41	0.02000	0.01510	0.00720	0.00002	0.04093	0.03997	0.01481	0.00003
42	0.02000	0.01396	0.00640	0.00001	0.04524	0.03903	0.01374	0.00002
43	0.02000	0.01271	0.00560	0.00001	0.05000	0.03758	0.01256	0.00001
44	0.02000	0.01133	0.00480	0.00000	0.05525	0.03538	0.01126	0.00000
45	0.02000	0.00983	0.00400	0.00000	0.06106	0.03236	0.00982	0.00000
46	0.02000	0.00818	0.00320	0.00000	0.06749	0.02840	0.00823	0.00000
47	0.02000	0.00639	0.00240	0.00000	0.07458	0.02335	0.00647	0.00000
48	0.02000	0.00443	0.00160	0.00000	0.08243	0.01705	0.00452	0.00000
49	0.02000	0.00231	0.00080	0.00000	0.09110	0.00934	0.00238	0.00000
50	0.02000	0.00000	0.00000	0.00000	0.10068	0.00000	0.00000	0.00000

TABLE 5. Discounted future service flow per capital unit as a function of age, $\phi_{\rho}(s)$.

$\phi_{o}(s)$ = Expected remaining life time.

Technical life time: N = 6, 20, and 50.

Interest rate: Rho (ρ) = 0 and 0.10.

N = 6

s	RHO = 0				RHO = 0.10			
	simult. exit	class II m=5	class I n=1	class I n=5	simult. exit	class II m=5	class I n=1	class I n=5
00	6.00000	5.00000	3.00000	1.00000	4.51188	3.91225	2.48018	0.92031
01	5.00000	4.00054	2.50000	0.83333	3.93469	3.27244	2.13060	0.77733
02	4.00000	3.01377	2.00000	0.66667	3.29680	2.57642	1.75799	0.63039
03	3.00000	2.08064	1.50000	0.50000	2.59182	1.85675	1.36060	0.47935
04	2.00000	1.25276	1.00000	0.33333	1.81269	1.16458	0.93654	0.32404
05	1.00000	0.55991	0.50000	0.16667	0.95163	0.54070	0.48374	0.16432
06	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

TABLE 5 (cont.)

N = 20

s	RHO = 0 simult. exit	RHO = 0 class II m=5	RHO = 0 class I n=1	RHO = 0 class I n=5	RHO = 0.10 simult. exit	RHO = 0.10 class II m=5	RHO = 0.10 class I n=1	RHO = 0.10 class I n=5
00	20.00000	16.66670	10.00000	3.33333	8.64664	8.02551	5.67664	2.57510
01	19.00000	15.66670	9.50000	3.16667	8.50431	7.81785	5.52401	2.47517
02	18.00000	14.66680	9.00000	3.00000	8.34701	7.58843	5.36273	2.37280
03	17.00000	13.66770	8.50000	2.83333	8.17316	7.33531	5.19222	2.26791
04	16.00000	12.67090	8.00000	2.66667	7.98103	7.05689	5.01182	2.16041
05	15.00000	11.67890	7.50000	2.50000	7.76870	6.75209	4.82083	2.05020
06	14.00000	10.69510	7.00000	2.33333	7.53403	6.42050	4.61851	1.93721
07	13.00000	9.72386	6.50000	2.16667	7.27468	6.06251	4.40406	1.82132
08	12.00000	8.77013	6.00000	2.00000	6.98806	5.67923	4.17659	1.70245
09	11.00000	7.83900	5.50000	1.83333	6.67129	5.27252	3.93517	1.58050
10	10.00000	6.93548	5.00000	1.66667	6.32120	4.84475	3.67877	1.45533
11	9.00000	6.06413	4.50000	1.50000	5.93430	4.39873	3.40632	1.32685
12	8.00000	5.22878	4.00000	1.33333	5.50671	3.93743	3.11661	1.19493
13	7.00000	4.43234	3.50000	1.16667	5.03415	3.46386	2.80836	1.05946
14	6.00000	3.67679	3.00000	1.00000	4.51188	2.98085	2.48019	0.92031
15	5.00000	2.96308	2.50000	0.83333	3.93469	2.49097	2.13061	0.77733
16	4.00000	2.29129	2.00000	0.66667	3.29680	1.99640	1.75800	0.63039
17	3.00000	1.66069	1.50000	0.50000	2.59182	1.49890	1.36060	0.47935
18	2.00000	1.06990	1.00000	0.33333	1.81269	0.99380	0.93653	0.32404
19	1.00000	0.51708	0.50000	0.16667	0.95163	0.50000	0.48374	0.16432
20	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

TABLE 5 (cont.)

N = 50

s	Rho = 0 simult. exit	Rho = 0 class II m=5	Rho = 0 class I n=1	Rho = 0 class I n=5	Rho = 0.10 simult. exit	Rho = 0.10 class II m=5	Rho = 0.10 class I n=1	Rho = 0.10 class I n=5
00	50.00000	41.66670	25.00000	8.33333	9.93262	9.78515	8.01352	4.73880
01	49.00000	40.66670	24.50000	8.16667	9.92553	9.76255	7.97442	4.68602
02	48.00000	39.66670	24.00000	8.00000	9.91770	9.73758	7.93385	4.63222
03	47.00000	38.66670	23.50000	7.83333	9.90904	9.70999	7.89172	4.57739
04	46.00000	37.66680	23.00000	7.66667	9.89948	9.67951	7.84797	4.52151
05	45.00000	36.66700	22.50000	7.50000	9.88891	9.64587	7.80249	4.46453
06	44.00000	35.66760	22.00000	7.33333	9.87722	9.60878	7.75520	4.40643
07	43.00000	34.66860	21.50000	7.16666	9.86431	9.56793	7.70599	4.34718
08	42.00000	33.67030	21.00000	7.00000	9.85004	9.52299	7.65477	4.28675
09	41.00000	32.67310	20.50000	6.83333	9.83427	9.47366	7.60141	4.22510
10	40.00000	31.67730	20.00000	6.66666	9.81684	9.41961	7.54580	4.16219
11	39.00000	30.68340	19.50000	6.50000	9.79758	9.36049	7.48781	4.09800
12	38.00000	29.69190	19.00000	6.33333	9.77629	9.29598	7.42730	4.03249
13	37.00000	28.70330	18.50000	6.16666	9.75276	9.22575	7.36412	3.96561
14	36.00000	27.71840	18.00000	6.00000	9.72676	9.14947	7.29813	3.89733
15	35.00000	26.73770	17.50000	5.83333	9.69802	9.06681	7.22914	3.82761
16	34.00000	25.76210	17.00000	5.66666	9.66626	8.97745	7.15698	3.75640
17	33.00000	24.79220	16.50000	5.50000	9.63117	8.88109	7.08146	3.68365
18	32.00000	23.82890	16.00000	5.33333	9.59238	8.77741	7.00238	3.60933
19	31.00000	22.87300	15.50000	5.16667	9.54951	8.66614	6.91951	3.53339
20	30.00000	21.92530	15.00000	5.00000	9.50213	8.54698	6.83262	3.45580
21	29.00000	20.96670	14.50000	4.83333	9.44977	8.41966	6.74145	3.37645
22	28.00000	20.05790	14.00000	4.66666	9.39190	8.28391	6.64575	3.29532
23	27.00000	19.13980	13.50000	4.50000	9.32794	8.13949	6.54520	3.21236

TABLE 5 (cont.)

N = 50 (cont.)

s	Rho = 0				Rho = 0.10			
	simult. exit	class II m=5	class I n=1	class I n=5	simult. exit	class II m=5	class I n=1	class I n=5
24	26.00000	18.23320	13.00000	4.33333	9.25726	7.98615	6.43951	3.12750
25	25.00000	17.33870	12.50000	4.16667	9.17915	7.82365	6.32833	3.04069
26	24.00000	16.45710	12.00000	4.00000	9.09282	7.65176	6.21132	2.95187
27	23.00000	15.58910	11.50000	3.83333	8.99741	7.47024	6.03807	2.86096
28	22.00000	14.73520	11.00000	3.66666	8.89197	7.27888	5.95819	2.76791
29	21.00000	13.89600	10.50000	3.50000	8.77543	7.07746	5.82121	2.67264
30	20.00000	13.07190	10.00000	3.33333	8.64664	6.86574	5.67667	2.57508
31	19.00000	12.26350	9.50000	3.16667	8.50431	6.64349	5.52404	2.47515
32	18.00000	11.47100	9.00000	3.00000	8.34701	6.41049	5.36276	2.37279
33	17.00000	10.69480	8.50000	2.83333	8.17316	6.16648	5.19225	2.26792
34	16.00000	9.93505	8.00000	2.66667	7.98103	5.91122	5.01184	2.16041
35	15.00000	9.19197	7.50000	2.50000	7.76870	5.64442	4.82086	2.05021
36	14.00000	8.46565	7.00000	2.33333	7.53403	5.36582	4.61854	1.93721
37	13.00000	7.75615	6.50000	2.16667	7.27468	5.07508	4.40403	1.82133
38	12.00000	7.06345	6.00000	2.00000	6.98806	4.77188	4.17661	1.70246
39	11.00000	6.38752	5.50000	1.83333	6.67129	4.45583	3.93518	1.58849
40	10.00000	5.72822	5.00000	1.66667	6.32120	4.12669	3.67878	1.45532
41	9.00000	5.08542	4.50000	1.50000	5.93430	3.78391	3.40632	1.32685
42	8.00000	4.45892	4.00000	1.33333	5.58671	3.42706	3.11650	1.19494
43	7.00000	3.84849	3.50000	1.16667	5.03415	3.05569	2.80335	1.05946
44	6.00000	3.25387	3.00000	1.00000	4.51183	2.66928	2.48819	0.92031
45	5.00000	2.67476	2.50000	0.83333	3.93469	2.26724	2.13060	0.77733
46	4.00000	2.11034	2.00000	0.66667	3.29680	1.84899	1.75799	0.63039
47	3.00000	1.56177	1.50000	0.50000	2.59182	1.41388	1.36060	0.47535
48	2.00000	1.02719	1.00000	0.33333	1.81269	0.96119	0.93653	0.32404
49	1.00000	0.50670	0.50000	0.16667	0.95163	0.49017	0.48374	0.16432
50	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

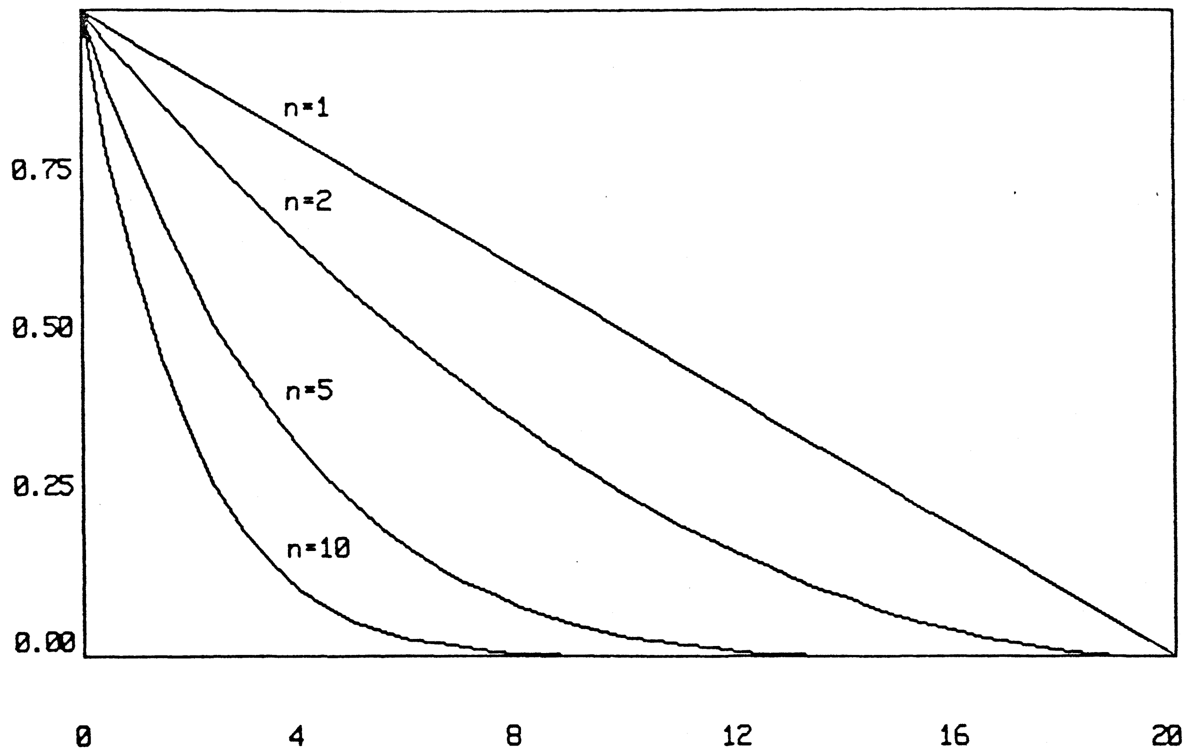


FIGURE 3. Survival functions for gross capital, B(s).
Class I. N=20.

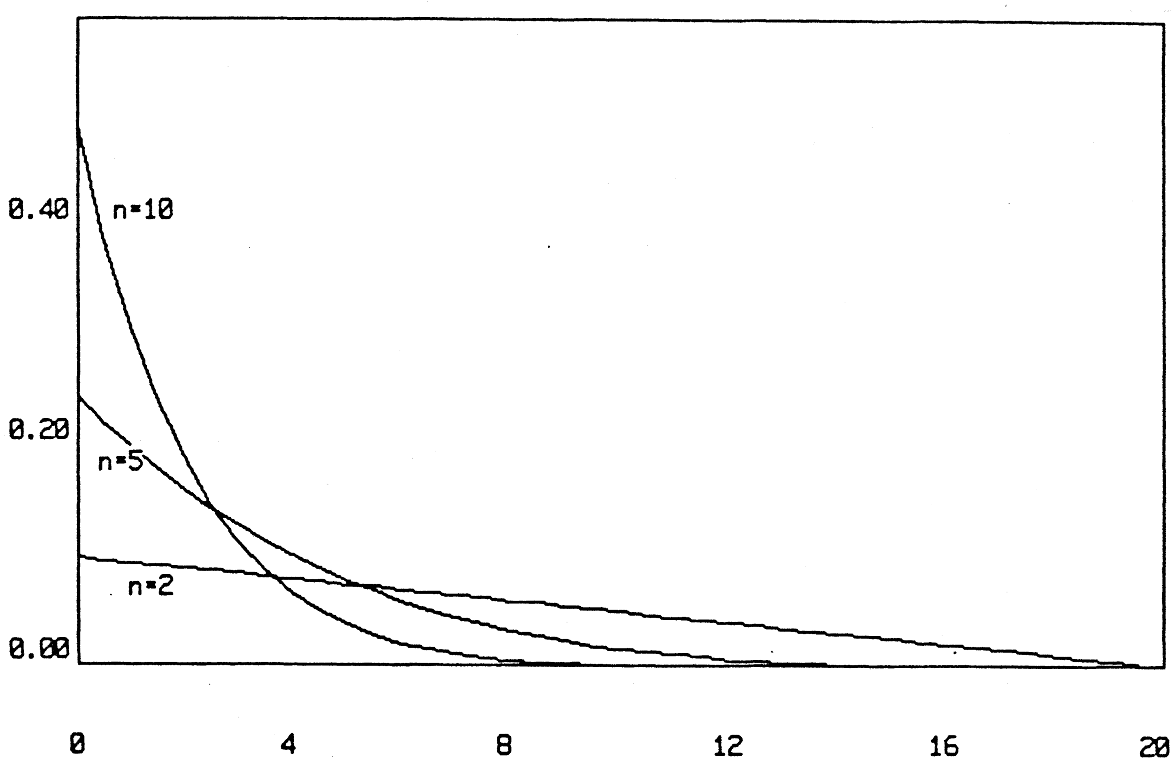


FIGURE 4. Replacement (density) functions, b(s).
Class I. N=20.

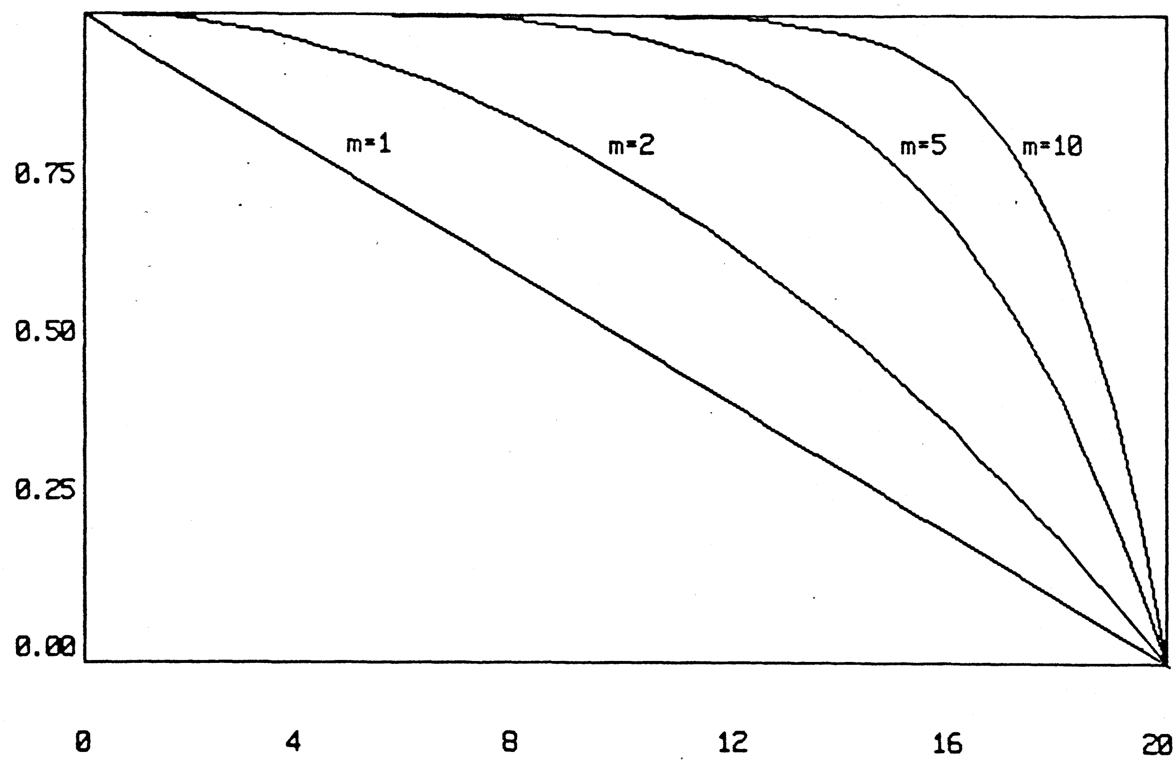


FIGURE 5. Survival functions for gross capital, B(s).
Class II. N=20.

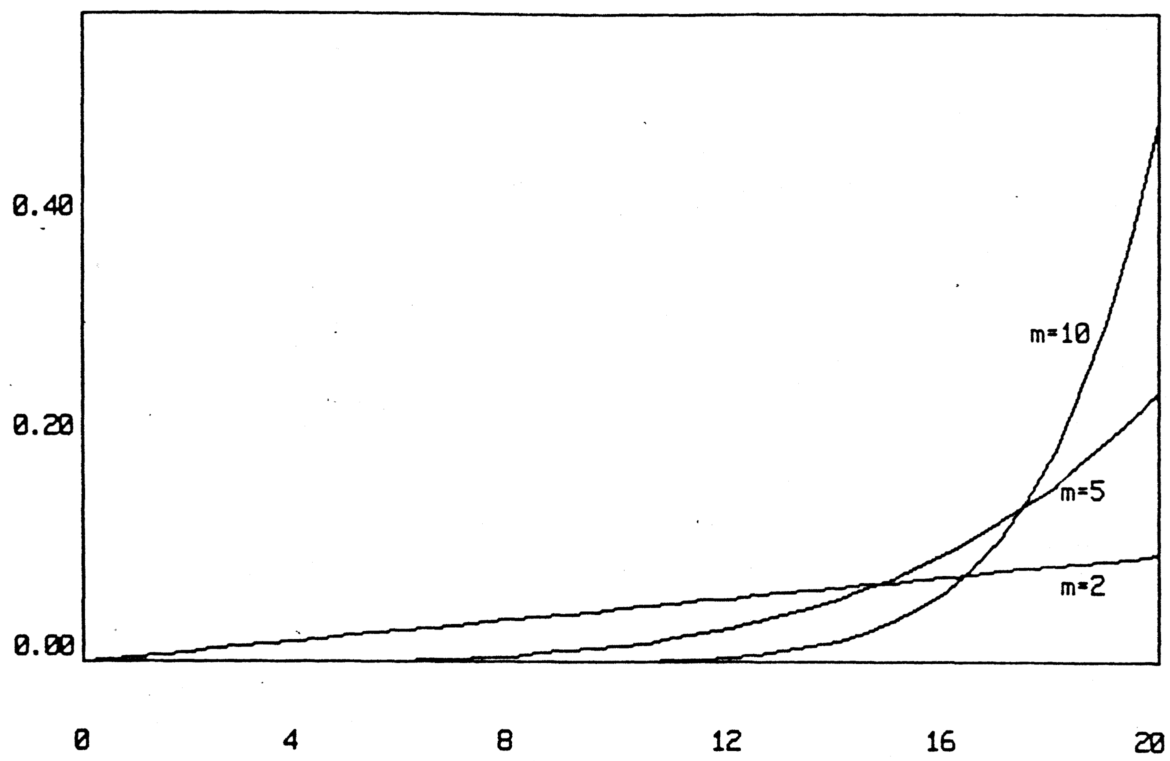


FIGURE 6. Replacement (density) functions, b(s).
Class II. N=20.

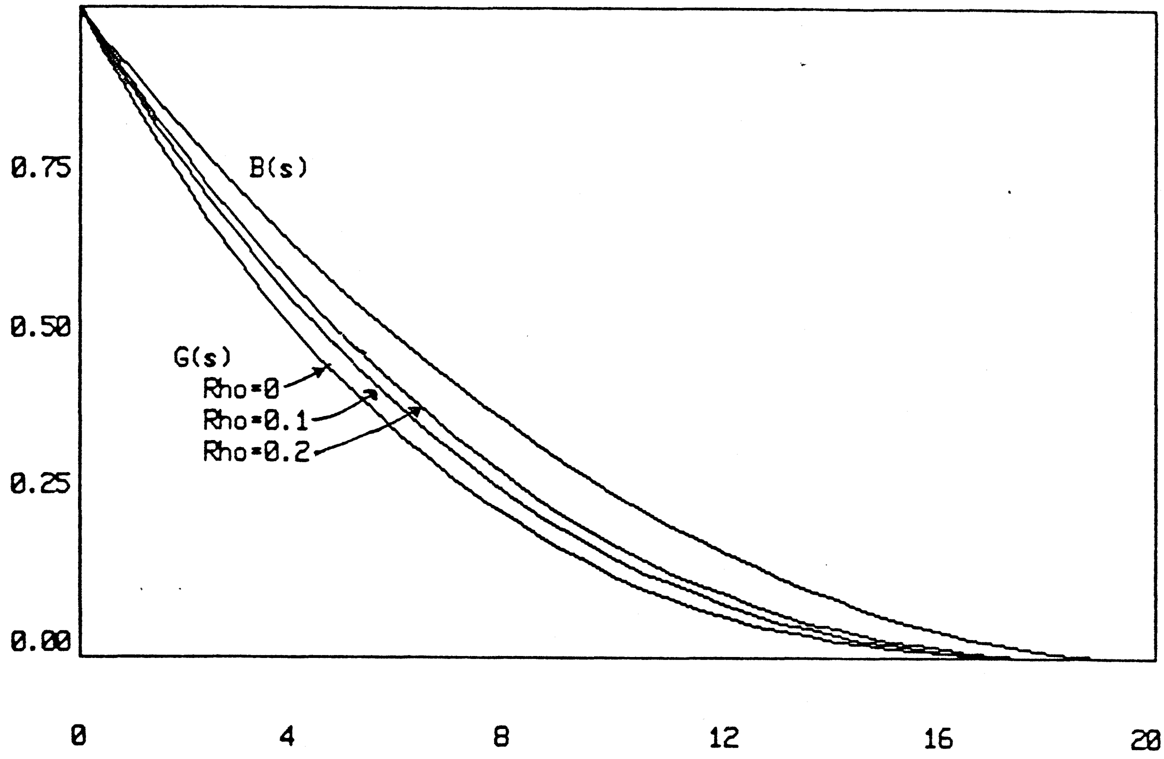


FIGURE 7. Survival function for gross capital, B(s), and corresponding weighting functions for net capital, G(s).
Class I. N=20, n=2.

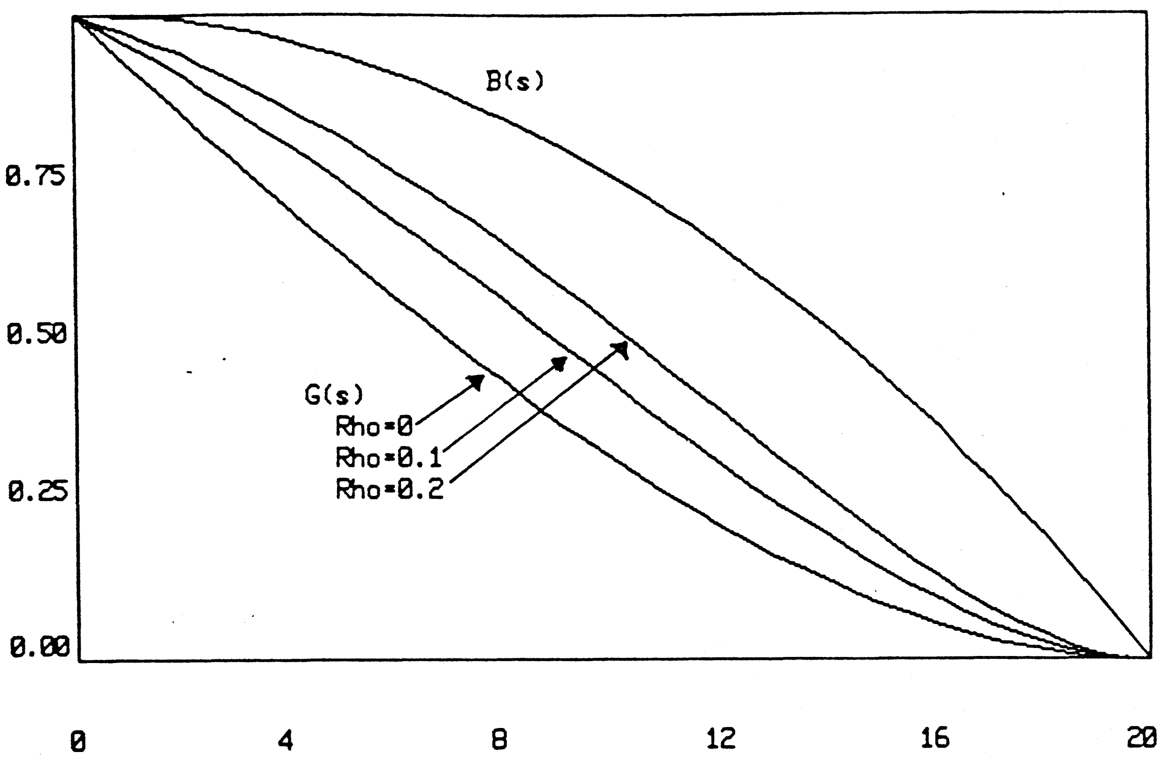


FIGURE 8. Survival function for gross capital, B(s), and corresponding weighting functions for net capital, G(s).
Class II. N=20, m=2.

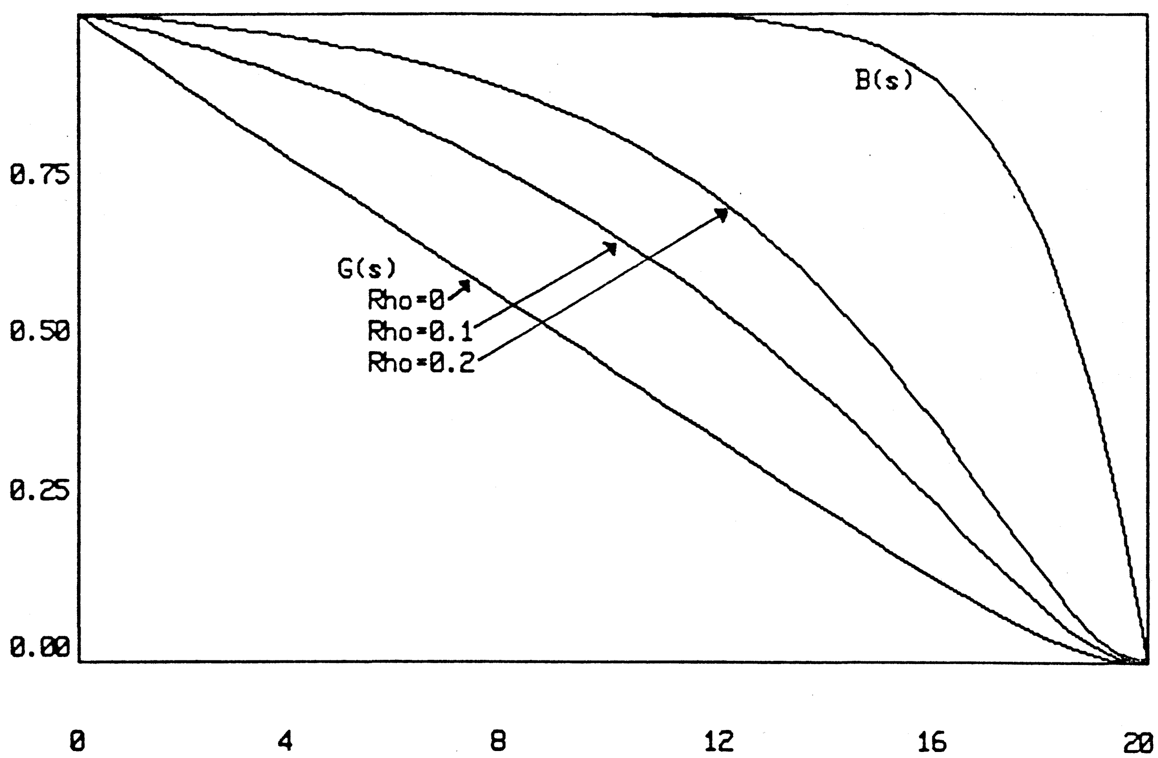


FIGURE 9. Survival function for gross capital, B(s), and corresponding weighting functions for net capital, G(s).
Class II. N=20, m=10.

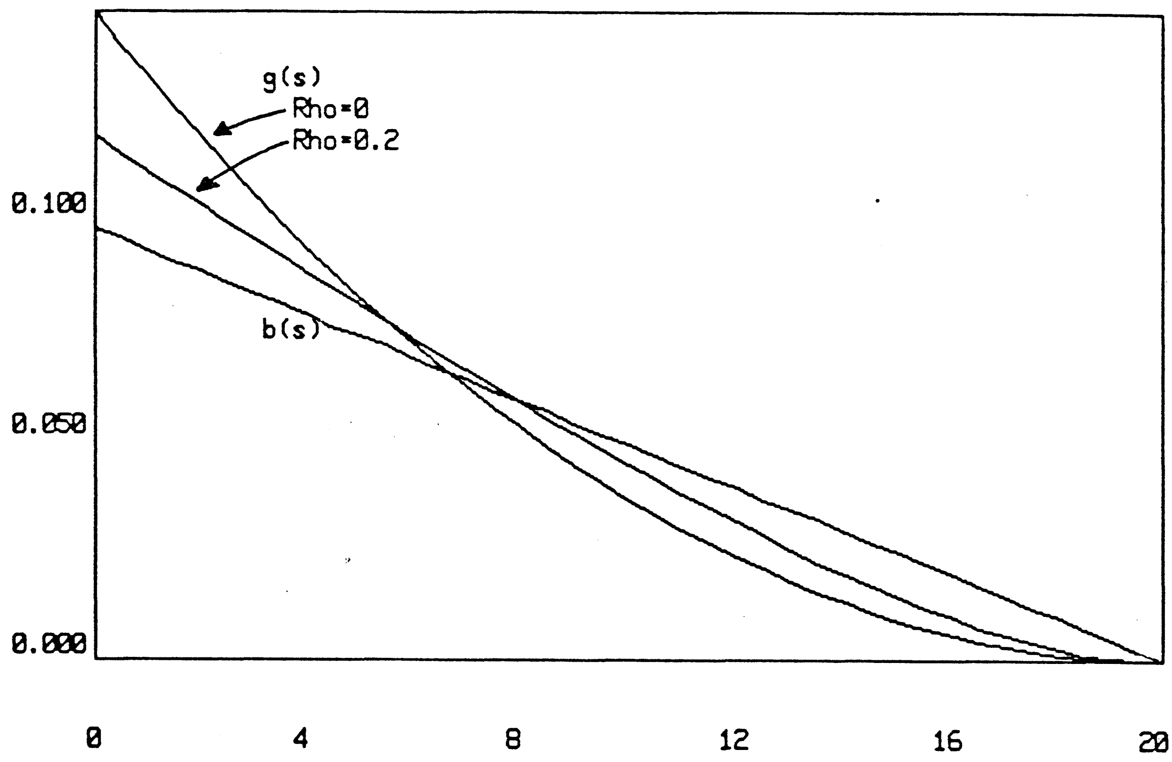


FIGURE 10. Replacement (density) function, b(s), and corresponding depreciation functions, g(s).
Class I. N=20, n=2.

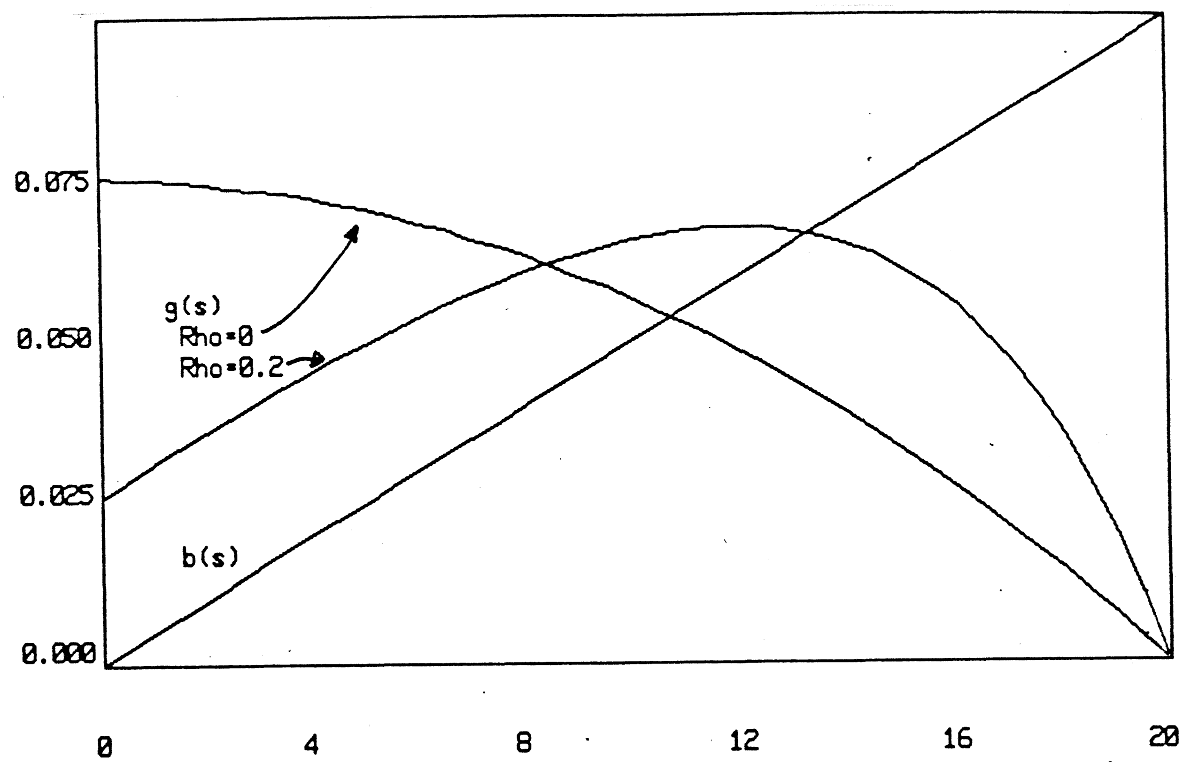


FIGURE 11. Replacement (density) function, b(s), and corresponding depreciation functions, g(s).
Class II. N=20, m=2.

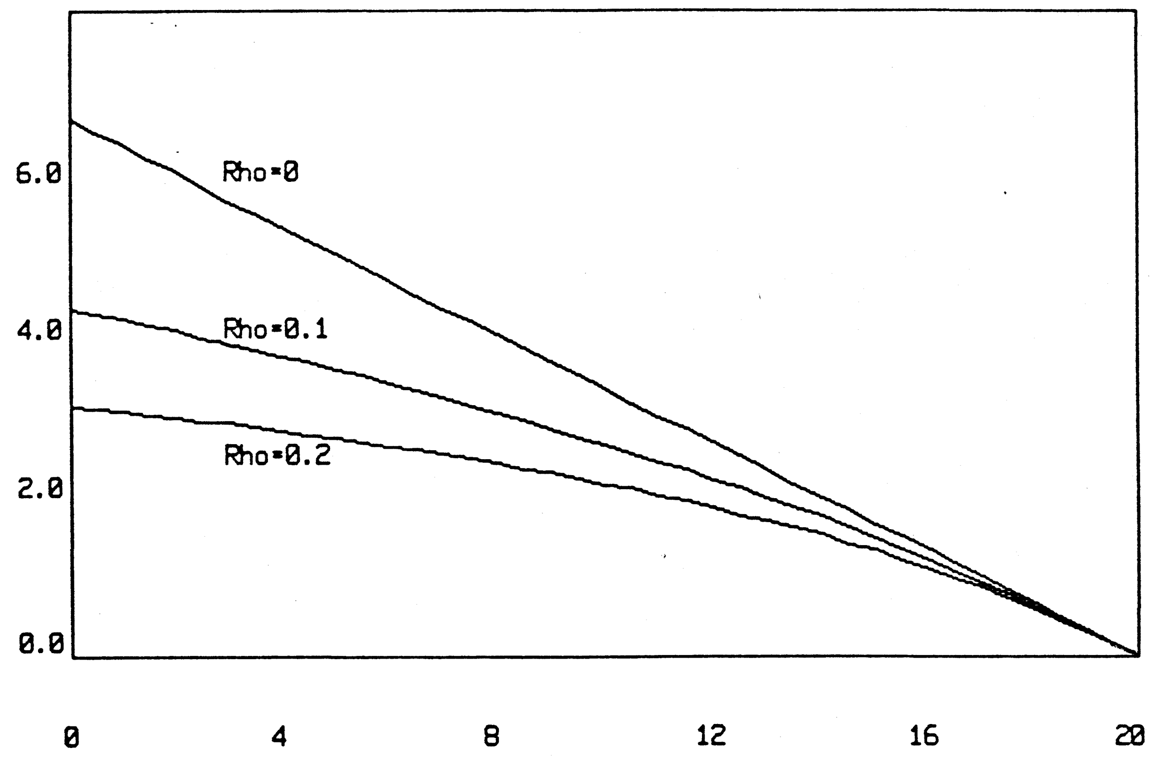


FIGURE 12. The function $\Phi(s)$.
Class I. N=20, n=2.

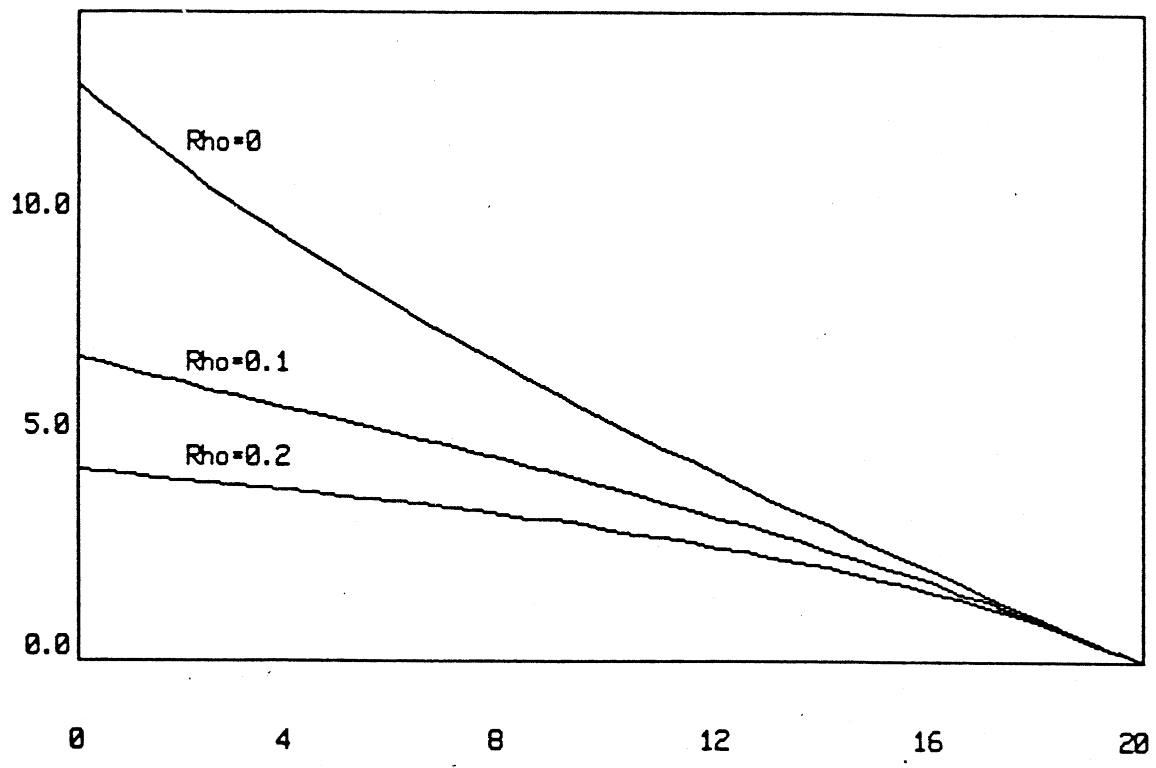


FIGURE 13. The function $\Phi(s)$.
Class II. N=20, m=2.

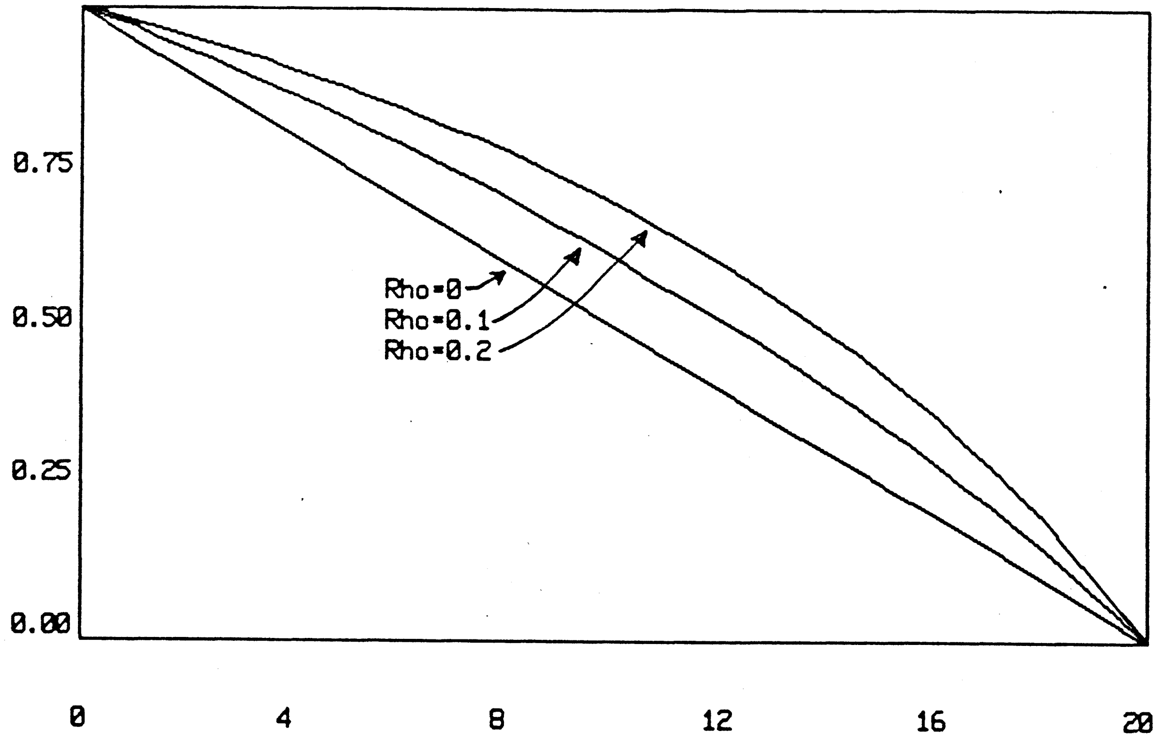


FIGURE 14. The function $\mu(s)=q(t,s)/q(t)$.
Class I. N=20, n=2.

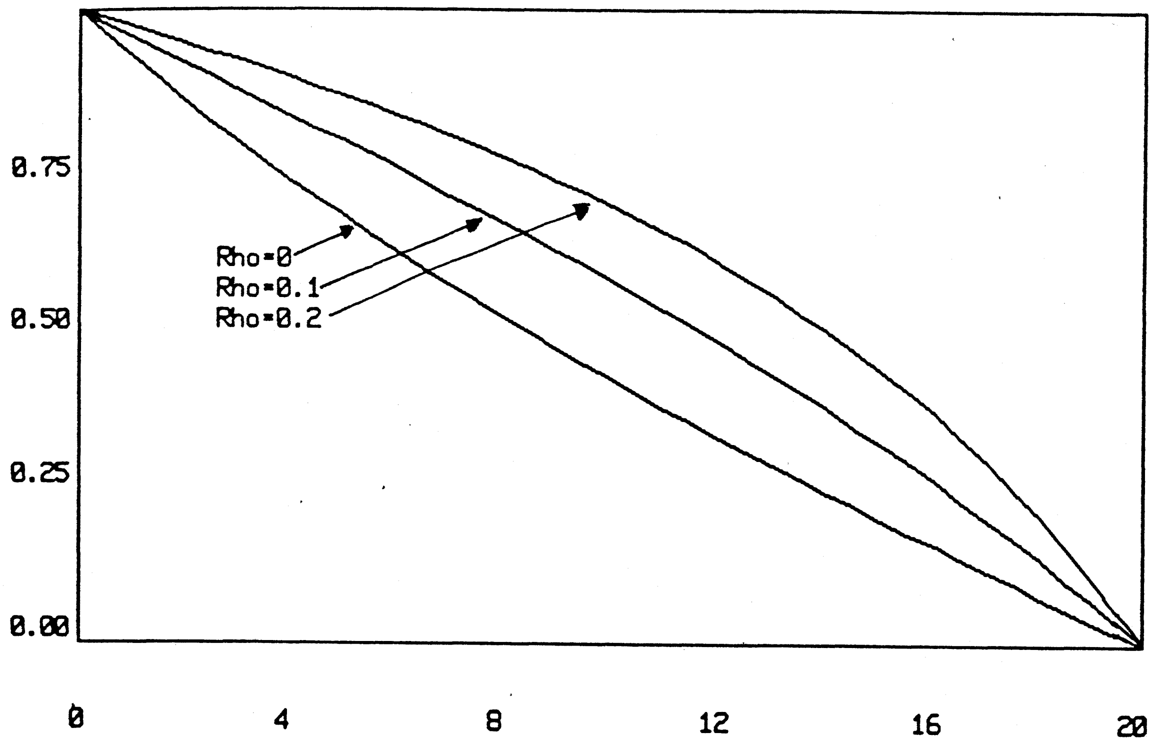


FIGURE 15. The function $\mu(s)=q(t,s)/q(t)$.
Class II. N=20, n=2.

We shall also briefly discuss two classes of two-parametric survival functions - one convex, one concave - which emerge as simple transformations of the standard exponential function (53).

CLASS III: Convex: Truncated exponential

A main problem with the standard exponential survival function $B(s)=e^{-\delta s}$ is its assumption of an infinite service life. In practice, the distribution must be truncated in some way. This motivates considering the following modification, in which maximal life time is restricted to be finite and equal to N:

$$(96) \qquad B(s) = B^{III}(s;N,\delta) = \begin{cases} \dfrac{e^{-\delta s}-e^{-\delta N}}{1-e^{-\delta N}} & \text{for } 0\le s\le N \\ 0 & \text{for } s>N, \end{cases}$$

where δ is positive. The corresponding replacement (density) function is

$$(97) \qquad b(s) = b^{III}(s;N,\delta) = \begin{cases} \dfrac{\delta e^{-\delta s}}{1-e^{-\delta N}} & \text{for } 0\le s\le N \\ 0 & \text{for } s>N. \end{cases}$$

It is convenient to reformulate the model in terms of the function $H_a(M) = (1-e^{-aM})/a$, which may be interpreted as the present value of a constant annuity of 1 over M years discounted at the rate a. This gives

$$(98) \qquad B(s) = e^{-\delta s}\frac{H_\delta(N-s)}{H_\delta(N)} ,$$

$$(99) \qquad b(s) = e^{-\delta s}\frac{1}{H_\delta(N)} \qquad 0\le s\le N.$$

Substituting these expressions in (9), while making use of (10), we get

$$(100) \qquad \Phi_\rho(s) = \frac{H_\delta(N-s)-H_{\rho+\delta}(N-s)}{\rho H_\delta(N-s)} ,$$

$$(101) \qquad \psi_\rho(s) = \frac{H_{\rho+\delta}(N-s)}{H_\delta(N-s)} .$$

Hence, using (33) and (41), we find that the weighting functions for net capital and depreciation are, respectively,

$$(102) \qquad G_\rho(s) = e^{-\delta s}\frac{H_\delta(N-s) - H_{\rho+\delta}(N-s)}{H_\delta(N) - H_{\rho+\delta}(N)} ,$$

$$(103) \qquad g_\rho(s) = e^{-\delta s}\frac{\rho H_{\rho+\delta}(N-s)}{H_\delta(N) - H_{\rho+\delta}(N)} .$$

These expressions are different from (98) and (99), i.e. in the truncated exponential model, the concepts gross and net capital do not - in contrast to those in the standard exponential model - coincide.

From (23) and (100) we find that expected remaining life time as a function of age is in this case

$$(104)\quad E(T;s) = \Phi_0(s) = \frac{1}{\delta} - \frac{N-s}{e^{\delta(N-s)}-1} \ ,$$

which is a decreasing function of δ and an increasing function of N. The last term in this expression shows the effect of the truncation of the distribution. This effect can also be read off from the expressions for the vintage prices and the capital service price. Inserting (100) in (27), (29), and (28), we get

$$(105)\quad q(t,s) = q(t)\,\frac{\{H_\delta(N-s) - H_{\rho+\delta}(N-s)\}H_\delta(N)}{\{H_\delta(N) - H_{\rho+\delta}(N)\}H_\delta(N-s)} \ ,$$

$$(106)\quad p(t,s) = q(t)e^{-\delta s}\,\frac{H_\delta(N-s) - H_{\rho+\delta}(N-s)}{H_\delta(N) - H_{\rho+\delta}(N)} \ ,$$

$$(107)\quad c(t) = q(t)\cdot\frac{\rho H_\delta(N)}{H_\delta(N) - H_{\rho+\delta}(N)} \ .$$

Since $H_a(M) \underset{M\to\infty}{\to} 1/a$, it is easy to verify that (100)-(107) give the same result as (55)-(57), (59)-(60), (63)-(65) when the maximal life time N goes to infinity.

Three particular cases of this class of survival functions are worth noting:

First, as already noted, when $N\to\infty$, the model converges towards the standard exponential model.

Second, when $\delta\to 0$, eq. (96) approaches $B(s) = 1 - s/N$, i.e. a linearly decreasing survival function, at the limit.

Third, when $\delta\to\infty$, the model degenerates to $B(0) = 1$, $B(s) = 0$ for $s>0$, i.e. a specification with instantaneous scrapping of the capital.

When $0<\delta<\infty$, the survival function and the retirement (density) function are both convex in this case, since

$$\frac{d^2B(s)}{ds^2} = \frac{\delta^2 e^{-\delta s}}{1-e^{-\delta N}} > 0$$

and

$$\frac{d^2b(s)}{ds^2} = \frac{\delta^3 e^{-\delta s}}{1-e^{-\delta N}} > 0.$$

The basic curvature of this class of functions is thus the same as in class I for $2<n<\infty$. There is, however, one notable difference: The retirement function b(s) is continuous at s=N in class I - since (67) implies b(N)=0 (whenever n>1) - whereas it is discontinuous in class III - since (97) implies $b(N) = \delta e^{-\delta N}/(1-e^{-\delta N}) > 0$.

CLASS IV: Concave: Inverse truncated exponential

We can generate a fourth class of two-parametric survival profiles by reversing the sign of the parameter δ in class III. This, of course, also implies a reversing of its curvature. Let $\gamma = -\delta$, where γ is defined to be positive. This gives the survival function

$$(108)\quad B(s) = B^{IV}(s;N,\gamma) = \begin{cases} \dfrac{e^{\gamma N}-e^{\gamma s}}{e^{\gamma N}-1} & \text{for } 0\le s\le N \\ 0 & \text{for } s>N, \end{cases}$$

with the corresponding replacement (density) function

$$(109)\quad b(s) = b^{IV}(s;N,\gamma) = \begin{cases} \dfrac{\gamma e^{\gamma s}}{e^{\gamma N}-1} & \text{for } 0\le s\le N \\ 0 & \text{for } s>N. \end{cases}$$

Since $H_{-a}(N) = e^{aN}H_a(N)$ we find directly, by substituting $\gamma = -\delta$ in (98)-(103),

$$(110)\quad B(s) = \frac{H_\gamma(N-s)}{H_\gamma(N)} ,$$

$$(111)\quad b(s) = e^{-\gamma(N-s)} \frac{1}{H_\gamma(N)} ,$$

$$(112)\quad \Phi_\rho(s) = \frac{H_\gamma(N-s)-e^{-\gamma(N-s)}H_{\rho-\gamma}(N-s)}{\rho H_\gamma(N-s)} ,$$

$$(113)\quad \psi_\rho(s) = e^{-\gamma(N-s)} \frac{H_{\rho-\gamma}(N-s)}{H_\gamma(N-s)} ,$$

$$(114)\quad G_\rho(s) = \frac{H_\gamma(N-s)-e^{-\gamma(N-s)}H_{\rho-\gamma}(N-s)}{H_\gamma(N)-e^{-\gamma N}H_{\rho-\gamma}(N)} ,$$

$$(115)\quad g_\rho(s) = e^{-\gamma(N-s)} \frac{\rho H_{\rho-\gamma}(N-s)}{H_\gamma(N)-e^{-\gamma N}H_{\rho-\gamma}(N)} .$$

Expected life time as a function of age is in this case

$$(116)\quad E(T;s) = \Phi_0(s) = \frac{N-s}{1-e^{-\gamma(N-s)}} - \frac{1}{\gamma} ,$$

and for the vintage prices and the capital service price we get, respectively,

$$(117)\quad q(t,s) = q(t) \frac{\{H_\gamma(N-s)-e^{-\gamma(N-s)}H_{\rho-\gamma}(N-s)\}\cdot H_\gamma(N)}{\{H_\gamma(N)-e^{-\gamma N}H_{\rho-\gamma}(N)\}\cdot H_\gamma(N-s)} ,$$

(118) $$p(t,s) = q(t)\,\frac{H_\gamma(N-s)-e^{-\gamma(N-s)}H_{\rho-\gamma}(N-s)}{H_\gamma(N)-e^{-\gamma N}H_{\rho-\gamma}(N)},$$

(119) $$c(t) = q(t)\,\frac{\rho H_\gamma(N)}{H_\gamma(N)-e^{-\gamma N}H_{\rho-\gamma}(N)}.$$

We note the following particular cases of this specification:

First: when $N\to\infty$, the model converges towards $B(s) = 1$ for all $s>0$, i.e. a specification with infinite service life and no deterioration of the capital.

Second, when $\gamma\to 0$, (108) degenerates to the linear function $B(s) = 1-s/N$.

Third, when $\gamma\to\infty$, we get the simultaneous exit specification ($B(s) = 1$ for $0\leq s\leq N$, 0 otherwise) as the limiting case.

When $0<\gamma<\infty$, the survival function is (strictly) concave and the retirement (density) function is convex in this case, since

$$\frac{d^2B(s)}{ds^2} = -\frac{\gamma^2e^{\gamma s}}{e^{\gamma N}-1} < 0,$$

and

$$\frac{d^2b(s)}{ds^2} = \frac{\gamma^3e^{\gamma s}}{e^{\gamma N}-1} > 0.$$

The curvature of this class of functions is thus basically the same as in class II for $2<m<\infty$.[19)] There is, however, one notable difference: In class II, we have $b(0) = 0$ (when $m>1$), whereas $b(0) = \gamma/(e^{\gamma N}-1)>0$ in class IV; i.e. in the former, the retirement starts at zero, in the latter, the initial retirement is positive.

Overview and a generalization

The relationship between the four classes of two-parametric survival functions presented above is illustrated in figure 16. We see that the linear function $B(s) = 1-s/N$ is a member of all these families of survival functions. The simultaneous exit model (one horse shay) emerges as a special case of class I, II, and IV. Furthermore, the standard exponential function $B(s) = e^{-\delta s}$ is a common member of class I and III, as is also the specification with instantaneous retirement of the capital. These parametrizations thus make it possible - if suitable data are available - to test whether the standard specifications of the retirement process (exponential, linear, simultaneous exit) are valid approximations or not.

Still more interesting from this point of view would be a model which contains all the four classes of survival profiles as special cases, and hence can be used as the basis for a multiple testing scheme. One such model is the following function with four parameters, N, λ, σ, μ:

---

19) Note that eqs. (66), (80), (96), and (108) imply the following *symmetry* between the models in class I and III on the one hand and those in class II and IV on the other

$$B^{II}(s;N,n) = 1-B^{I}(N-s;N,n),$$

$$B^{IV}(s;N,\delta) = 1-B^{III}(N-s;N,\delta).$$

Thus, for each model in class II we can find a corresponding model in class I, and for each model in class IV we can find a corresponding model in class III, and vice versa. It is easy to show that two models which are symmetric in this sense have the general property that the sum of their expected life times is equal to the maximal life time N.

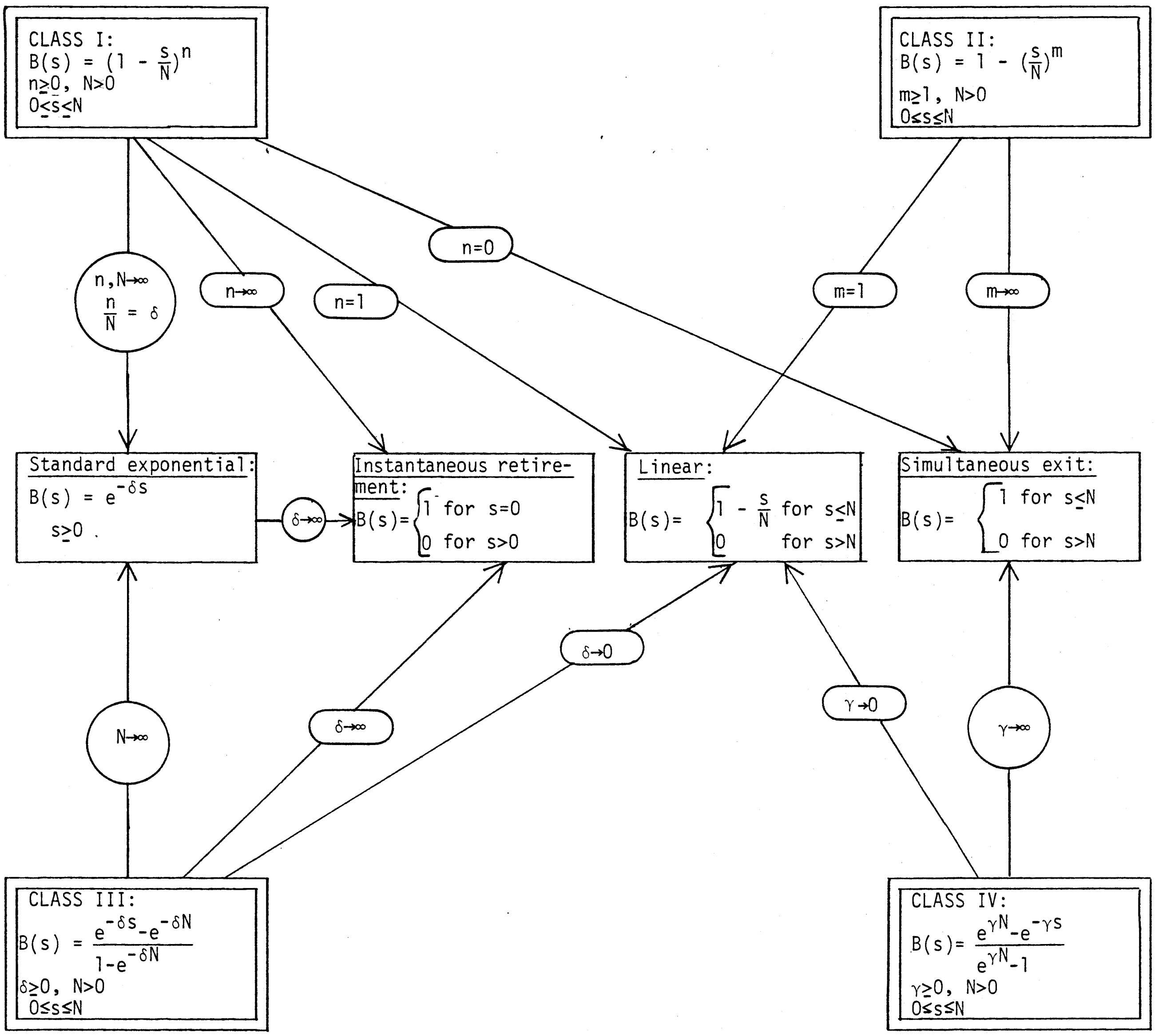


FIGURE 16. The relationship between some two-parametric survival functions.

$$(120)\qquad B(s) = \frac{\left(\lambda + \frac{N}{\sigma}\right)^{\mu} - \left(\lambda + \frac{s}{\sigma}\right)^{\mu}}{\left(\lambda + \frac{N}{\sigma}\right)^{\mu} - \lambda^{\mu}} \qquad \begin{array}{l} 0 \leq s \leq N; \\ \lambda \geq 0, \\ 0 < \mu < \infty, \\ -\infty < \sigma < \infty. \end{array}$$

We note that

Class I corresponds to: $\lambda=1$, $\sigma=-N$, $\mu=n$.

Class II corresponds to: $\lambda=0$, $\sigma=N$, $\mu=m$.

Class III corresponds to: $\lambda=1$, $\mu=-\delta\sigma$, $\sigma\to-\infty$.

Class IV corresponds to: $\lambda=1$, $\mu=\gamma\sigma$, $\sigma\to\infty$.

This function could be used, for instance, to test specifically for convexity or concavity of the retirement process by testing the parameters μ and σ. Needless to say, such a model would place strong claims on data.

Appendix A

THE PRICE INTERPRETATION OF DEPRECIATION

In this appendix we show that the quantity interpretation of depreciation, as given in eqs. (38) and (42), has a 'dual' price interpretation.

The capital value can, after substituting $\Theta=t-s$ in (26), be written as

$$\text{(A.1)} \quad V(t) = \int_{-\infty}^{t} p(t,t-\Theta)J(\Theta)d\Theta,$$

where $p(t,t-\Theta) = q(t)\int_{t-\Theta}^{\infty} e^{-\rho(z-t+\Theta)}B(z)dz/\int_{0}^{\infty}e^{-\rho z}B(z)dz$, which follows from (29). Hence,

$$\text{(A.2)} \quad \dot{V}(t) = p(t,0)J(t) + \int_{-\infty}^{t} \frac{dp(t,t-\Theta)}{dt} J(\Theta)d\Theta$$

$$= q(t)J(t) + \int_{-\infty}^{t} \frac{\partial p(t,t-\Theta)}{\partial t} J(\Theta)d\Theta + \int_{-\infty}^{t} \frac{\partial p(t,t-\Theta)}{\partial (t-\Theta)} J(\Theta)d\Theta,$$

which gives the following expression for E(t)

$$\text{(A.3)} \quad E(t) = q(t)J(t) - \dot{V}(t) = -\left[\int_{-\infty}^{t} \frac{\partial p(t,t-\Theta)}{\partial t} J(\Theta)d\Theta + \int_{-\infty}^{t} \frac{\partial p(t,t-\Theta)}{\partial (t-\Theta)} J(\Theta)d\Theta\right].$$

We have now written the value of depreciation in terms of the change of the prices of old capital units. This change has two parts, corresponding to the two components of the total derivative $dp(t,t-\Theta)/dt$. The first,

$$\text{(A.4)} \quad \frac{\partial p(t,t-\Theta)}{\partial t} = \dot{q}(t) \frac{\int_{t-\Theta}^{\infty} e^{-\rho(z-t+\Theta)}B(z)dz}{\int_{0}^{\infty} e^{-\rho z}B(z)dz} = \dot{q}(t)G_{\rho}(t-\Theta),$$

where G_{ρ} is defined as in (33), represents the increase in the prices of all vintages of old capital goods which accompany the increase in the price of new capital goods. The second component,

$$\text{(A.5)} \quad \frac{\partial p(t,t-\Theta)}{\partial (t-\Theta)} = -q(t) \frac{B(t-\Theta)-\rho \int_{t-\Theta}^{\infty} e^{-\rho(z-t+\Theta)}B(z)dz}{\int_{0}^{\infty} e^{-\rho z}B(z)dz} \doteq -q(t)g_{\rho}(t-\Theta),$$

where g_{ρ} is defined as in (41), is the 'cohort component'. It represents the decline in the vintage prices with increasing age. This component reflects the fact that all capital objects become gradually older and therefore yield a gradually decreasing flow of prospective capital services. Combining (A.4) and (A.5) with (32) and (42), we get

$$\text{(A.6)} \quad \int_{-\infty}^{t} \frac{\partial p(t,t-\Theta)}{\partial t} J(\Theta)d\Theta = \dot{q}(t)K_N(t),$$

and

$$\text{(A.7)} \quad \int_{-\infty}^{t} \frac{\partial p(t,t-\Theta)}{\partial (t-\Theta)} J(\Theta)d\Theta = -q(t)D_N(t).$$

Eqs. (A.6) and (A.7) show that the decomposition of E(t) given in (A.3) agrees with that in (39). The former interprets depreciation "from the price side", the latter interprets it "from the quantity side". We may say that they are 'dual' interpretations.[1)]

1) The latter interpretation agrees with that of Hall (1968, pp. 40-41) and Jorgenson (1974, pp. 205-207), who, following Hotelling (1925), define depreciation as the decline in prices of used capital goods over time. From their definitions, however, we cannot establish relationships between the quantity and price concepts similar to eqs. (36), (42), (43), (A.6), and (A.7) above, since Hall and Jorgenson avoid using a counterpart to our concept net capital in their analysis.

Appendix B

PROOF OF THE RECURRENCE FORMULAE (70) AND (84)

In this appendix, we give proofs of the two recurrence formulae used in section 6 when discussing class I and II of parametric survival profiles.

Class 1. We define the auxiliary function $C_\rho^I(s;N,n)$ as

$$C_\rho^I(s;N,n) = \int_s^N e^{-\rho(z-s)}(1-\frac{z}{N})^n dz. \tag{B.1}$$

Using integration by parts, we obtain

$$C_\rho^I(s;N,n) = \Big|_s^N -\frac{1}{\rho} e^{-\rho(z-s)}(1-\frac{z}{N})^n - \int_s^N\{-\frac{1}{\rho} e^{-\rho(z-s)}\}\cdot(-\frac{n}{N}(1-\frac{z}{N})^{n-1})\, dz .$$

The latter integral is equal to $n/(N\rho)\cdot C_\rho^I(s;N,n-1)$.

Hence,

$$C_\rho^I(s;N,n) = \frac{1}{\rho}(1-\frac{s}{N})^n - \frac{n}{N\rho}\cdot C_\rho^I(s;N,n-1), \qquad \rho>0;\ n=1,2,\ldots;\ 0\le s\le N. \tag{B.2}$$

The initial value for recursive application of (B.2) is

$$C_\rho^I(s;N,0) = \int_s^N e^{-\rho(z-s)}dz = \frac{1}{\rho}[1-e^{-\rho(N-s)}]. \quad \square \tag{B.3}$$

Class II. The auxiliary function $C_\rho^{II}(s;N,m)$ is defined as

$$C_\rho^{II}(s;N,m) = \int_s^N e^{-\rho(z-s)}\,(\frac{z}{N})^m dz. \tag{B.4}$$

Using integration by parts, we find

$$C_\rho^{II}(s;N,m) = \Big|_s^N -\frac{1}{\rho} e^{-\rho(z-s)}(\frac{z}{N})^m - \int_s^N\{-\frac{1}{\rho} e^{-\rho(z-s)}\}\cdot\frac{m}{N}(\frac{z}{N})^{m-1} dz.$$

The latter integral is equal to $-m/(N\rho)C_\rho^{II}(s;N,m-1)$.

Hence,

$$C_\rho^{II}(s;N,m) = \frac{1}{\rho}[(\frac{s}{N})^m - e^{-\rho(N-s)}] + \frac{m}{N\rho}C_\rho^{II}(s;N,m-1), \qquad \rho>0;\ m=1,2,\ldots;\ 0\le s\le N. \tag{B.5}$$

The initial value for recursive application of this formula is

$$C_\rho^{II}(s;N,0) = \int_s^N e^{-\rho(z-s)}dz = \frac{1}{\rho}[1-e^{-\rho(N-s)}]. \quad \square \tag{B.6}$$

Thus, the two recurrence formulae have the same initial value.

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