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OM BRUKEN AV LASPEYRES OG PAASCHE INDEKSER I EN NEOKLASSISK IMPORTMODELL

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P R E F A C E

In the Norwegian planning models developed by the Central Bureau of Statistics the bulk of imports is determined by means of an import share matrix of constant coefficients with exogenous adjustments. Recent work has attempted to endogenize these coefficient changes by making them functions of the relative price of imports.

The introduction of such an explicit functional form, describing the substitution possibilities between the imported and the domestic input, raises the question of whether one should continue to insist on the fact that the volume share of imported and domestic input should add to one, and what the consequences of such a normalization would be. The paper attempts to answer these questions first at the level of the input to a single sector, and then in the context of a multi-sectoral model as a whole.

Central Bureau of Statistical, Oslo, 10 April 1983

Arne Øien

F O R O R D

De norske planleggingsmodellene bruker en importandelsmatrise for å bestemme importnivået. Elementene i denne matrisen er, med unntak av mulige eksogene endringer, antatt konstante. I den senere tid har en arbeidet med å endogenisere disse andelene ved å la dem bli funksjoner av den relative importprisen.

Innføringen av slike eksplisitte funksjonsformer, som tillater substitusjon mellom importert og norskprodusert vareinnsats, fremtvinger spørsmålet om man bør fortsette å kreve at volumandelene for importert og hjemmeprodusert vareinnsats skal summere seg til én, og hva følgene av en slik normalisering vil være. Denne rapporten prøver å besvare disse spørsmålene, først ved å se på vareinnsatsen til en enkelt sektor, og deretter innenfor rammen av en fullstendig flersektor modell.

Statistisk Sentralbyrå, Oslo, 10. april 1983

Arne Øien

Central Bureau of Statistics
Research Department
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CONTENTS

INNHOLD

1. Introduction

This paper will analyze the consequences of using inconsistent Laspeyres and Paasche indices when modelling the behaviour of import shares in a simultaneous, multisectoral model.

In earlier works [see Frenger(1979a, 1979b, 1980)] we have used neoclassical production theory to explain the relationship between domestically produced and imported inputs. Total input of a commodity into a sector, f.ex. input of the i'th commodity into the k'th sector, is there defined by the neoclassical production function: it is no longer the sum (measured in constant prices) of the imported and the domestically produced input. In practice, however it may be convenient to insist on this adding up property, as is, for example, done in the national accounts. This is equivalent to defining the total input of commodity i to sector k as a Laspeyres quantity aggregate, while the implicitly defined price index becomes a Paasche aggregate. The purpose of this paper is to study the consequences of using these Laspeyres quantity aggregates and Paasche price aggregates when the true relationship between imported and domestically produced inputs is described by a neoclassical production function.

Let x_{ik}^{A} and x_{ik}^{B} be the quantity of imported and domestically produced input i delivered to sector k. The substitutability between the two inputs is described by the linear homogenous production function

$$x_{ik} = f^{ik}(x_{ik}^{A}, x_{ik}^{B}) \quad , \tag{1.1}$$

where x_{ik} represents the "total input" of commodity i into sector k. If additionally the prices of the imported and the domestically produced inputs are p_i^B and p_i^A, respectively, and the producer minimizes cost, then the substitutability between the two inputs is equivalently characterized by the dual unit cost function

$$p_{ik} = c^{ik}(p_i^A, p_i^B) \quad . \tag{1.2}$$

In applied situations it may be convenient to assume that f^{ik} and c^{ik} are CES functions, but most of the following analysis will be conducted in terms of arbitrary neoclassical functions. Applying Shephard's lemma to the unit cost function (1.2) gives the domestic coefficient and import coefficient functions:

$$\begin{aligned} m_{ik}^A(p_i^A,p_i^B) &= \frac{\partial}{\partial p_i^A} c^{ik}(p_i^A,p_i^B) = \frac{x_{ik}^A(p_i^A,p_i^B)}{x_{ik}} , \\ m_{ik}^B(p_i^A,p_i^B) &= \frac{\partial}{\partial p_i^B} c^{ik}(p_i^A,p_i^B) = \frac{x_{ik}^B(p_i^A,p_i^B)}{x_{ik}} , \end{aligned} \tag{1.3}$$

which may be interpreted as the unit factor demand equations.

In the empirical work [Frenger (1979a, 1980)] we used this theoretical framework to estimate the ratio x_{ik}^A/x_{ik}^B. The assumption of a CES technology with δ_{ik} as distribution parameter and σ_{ik} as the elasticity of substitution, gave us the relationship

$$\ln \frac{x_{ik}^A}{x_{ik}^B} = \ln \frac{1-\delta_{ik}}{\delta_{ik}} - \sigma_{ik} \ln \frac{p_i^A}{p_i^B} , \tag{1.4}$$

which is linear in the relative prices, and independent of the unknown "total input" x_{ik}.[1)]

1) See Frenger (1980), p. 13

The coefficient functions m^A_{ik} and m^B_{ik} will satisfy the production function $f^{ik}(m^A_{ik}, m^B_{ik}) = 1$, but we will in general have $m^A_{ik} + m^B_{ik} > 1$ for other than base year prices. But we can normalize these coefficients by defining the domestic shares and import shares measured in constant (base year) prices:

$$
\begin{aligned}
M^A_{ik}(p^A_i, p^B_i) &= \frac{m^A_{ik}(p^A_i, p^B_i)}{m^A_{ik}(p^A_i, p^B_i) + m^B_{ik}(p^A_i, p^B_i)} , \\
M^B_{ik}(p^A_i, p^B_i) &= \frac{m^B_{ik}(p^A_i, p^B_i)}{m_{ik}(p^A_i, p^B_i) + m^B_{ik}(p^A_i, p^B_i)} .
\end{aligned}
\tag{1.5}
$$

It follows from the definition that

$$M^A_{ik} + M^B_{ik} = 1 .$$

The domestic and import share functions define the unit isoquant of the Laspeyres output aggregates, and implicitly they define the Paasche price index

$$q_{ik} = \frac{p^A_i m^A_{ik} + p^B_i m^B_{ik}}{m^A_{ik} + m^B_{ik}} . \tag{1.6}$$

The next section will analyze the consequences of using inconsistent aggregates as applied to the input of an arbitrary commodity, domestically produced and imported, to a single sector. Most of the analysis will be in terms of an arbitrary neoclassical production function, but we will use a CES technology to illustrate our conclusions. In section 3 we present a neoclassical import model based on Frenger (1979b), and then use this model in section 4 of draw conclusions about the use of inconsistent aggregates in a complete model.

2. Import shares and Paasche price indices

This section will analyze in greater detail the import share functions $M^B(p^A,p^B)$ and the Paasche price aggregates $q(p^A,p^B)$, [see (1.5) and (1.6) respectively], and since we will be concerned with the flow of a single commodity to a single sector we will for the duration of this section ignore the commodity and sector subscripts.

2.1 Laspeyres import share indices

The import share indeces M^A and M^B may be considered as a pair of Laspeyres quantity aggregates, given that the base year prices are identically one (p^A=p^B=1 in the base year):[1)]

$$M^A(p^A,p^B) = \frac{m^A(p^A,p^B)}{m^A(p^A,p^B) + m^B(p^A,p^B)} , \tag{2.1.1}$$

$$M^B(p^A,p^B) = \frac{m^B(p^A,p^B)}{m^A(p^A,p^B) + m^B(p^A,p^B)} , \tag{2.1.2}$$

The import coefficients m^A and m^B are of course determined by the neoclassical technology described by $c(p^A,p^B)$ [see (1.2)]. It follows from the concavity of the production function that

$$m^A(p^A,p^B) + m^B(p^A,p^B) \geq 1 , \tag{2.1.3}$$

and thus that

---

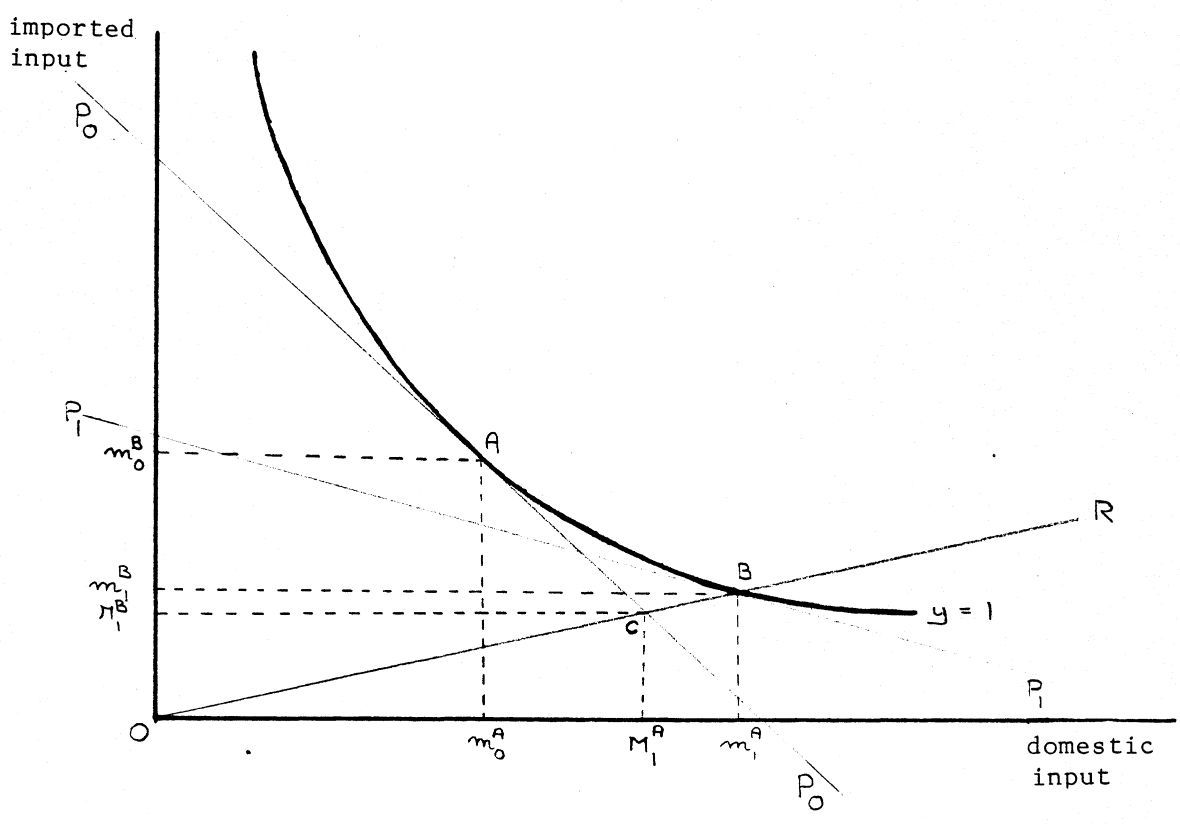
1) See eqs. (1.5) above.

$$M^A(p^A,p^B) \leq m^A(p^A,p^B) \quad ,$$
$$M^B(p^A,p^B) \leq m^B(p^A,p^B) \quad . \qquad (2.1.4)$$

This inequality is illustrated graphically in fig. 2.1.2 below.

It seems difficult to interpret M^A and M^B as factor demand equations derived from an underlying production function via cost minimization. They can probably best be interpreted as approximations, and this approximation may be illustrated by use of the following figure. Let y=1 represent the unit isoquant and let P_0-P_0 represent the (relative) prices of the base year. P_0-P_0 will

Fig. 2.1.1 - Derivation of the share functions M^A and M^B.



form a 45^{o} angle with the axis since prices are normalized to unity in the base year. The line P_0-P_0 represents the set of points whose sum equals one. The base year coefficients m_0^A and m_0^B are represented by the point A. Due to the base year normalization we also have that $m_0^A = M_0^A$, $m_0^B = M_0^B$ and $m_0^A + m_0^B = 1$.

Assume now that the price of imports increases and that the new relative prices are represented by the slope of P_1-P_1. Production will shift to point B and the new coefficients will be m_1^A and m_1^B. But, as we see from the diagram, $m_1^A+m_1^B > 1$. We now determine the shares M_1^A and M_1^B by reducing m_1^A and m_1^B proportionately by a factor Θ, so that $\Theta m^A + \Theta m^B = 1$. This Θ is given by $\Theta = (m_1^A+m_1^B)^{-1}<1$, and it gives:

$$M_1^A = \Theta\, m_1^A = \frac{m_1^A}{m_1^A + m_1^B} ,$$

$$M_1^B = \Theta\, m_1^B = \frac{m_1^B}{m_1^A + m_1^B} .$$

This proportionate reduction is represented by the line O-R, and the new import shares are given by the point C. This point C has to lie on the line P_0-P_0, which represents the set of points whose sum is one.

It should be pointed out that the location of the point C depends on the elasticity of substitution, i.e. on the curvature of the isoquant. A different σ would, with the same change in relative prices, have given us a different point C on the P_0-P_0 line.

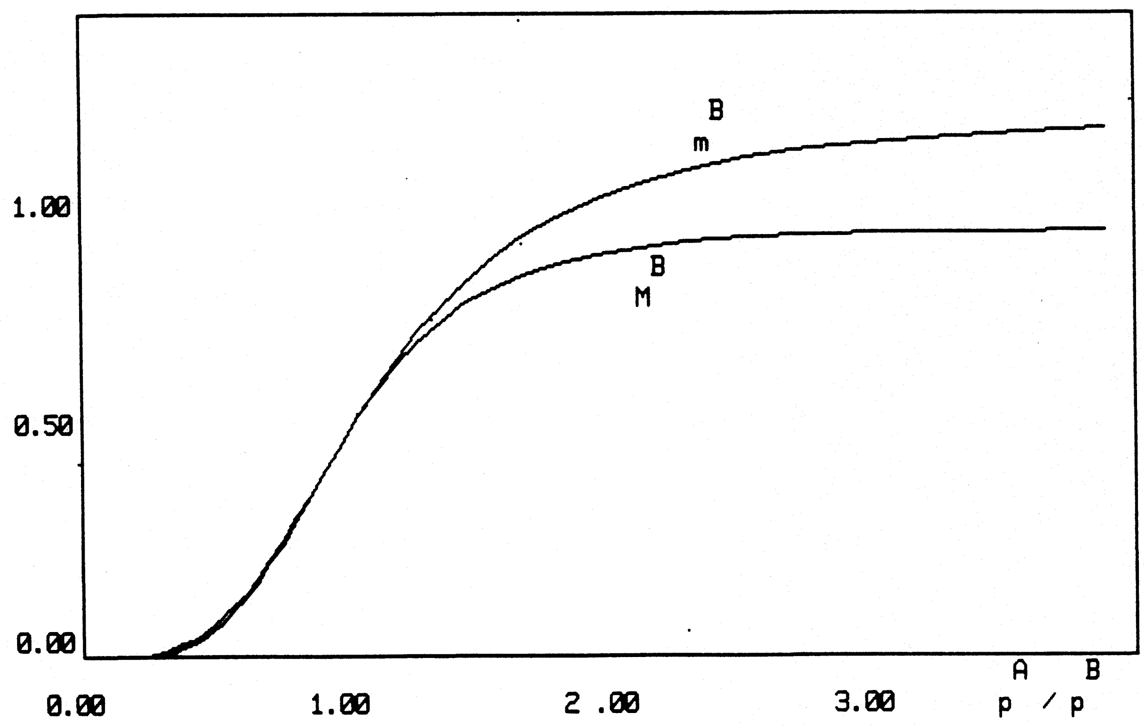
We may look upon $M(p^A,p^B) = [M^B(p^A,p^B), M^A(p^A,p^B)]'$ as a mapping from the positive orthant R_+^2 into the 2-dimensional simplex

$S^2 = \{x \varepsilon R^2_+ | x_1 + x_2 = 1\}$. The functions M^B and M^A are homogeneous of degree zero in prices, and may thus be expressed as functions of the price ratio p^A/p^B. Differentiating M^B with respect to this price ratio gives[2]

$$\frac{\partial M^B}{\partial (p^A/p^B)} = \frac{p^B}{p^A} \Theta^2(p^A,p^B) \; c_{AB}(p^A,p^B) \; c(p^A,p^B) \geq 0 \; . \qquad (2.1.5)$$

Since c_{AB}, the second derivative of the cost function, is positive, it follows that M^B is a monotone non-decreasing function of the price ratio p^A/p^B. This is illustrated graphically in fig. 2.1.2:

Fig. 2.1.2 The import share M^B and the import coefficient m^B



1) Using a CES function with $\delta = 0.5$ and $\sigma = 1$.

---

2) See (2.2.2) below for a formal definition of Θ.

We have implicitly been assuming that $c(p^A,p^B)$ is twice continuously differentiable. This ensures that m^A and m^B exist and are continuous functions.[3)] Thus M^A and M^B are continuous. The mapping M need not be onto the simplex S^2: the Leontief cost function, for example, is continuously differentiable, but M^A and M^B are constant functions. The mapping will be onto S^2 if and only if both inputs are non-essential.

From (2.1.5) it follows that M^B may be considered as a first order approximation at the base point, since we there have that

$$\frac{\partial M^B}{\partial(p^A/p^B)} = c_{AB} = \frac{\partial m^B}{\partial(p^A/p^B)} \quad . \tag{2.1.6}$$

This is illustrated graphically by the tangency between m^B and M^B at $p^A/p^B = 1$ in fig. 2.1.2. In general, however, the inequality (2.1.4) will hold.

---

3) Formally, this has the unfortunate consequence of excluding the case where $\sigma = \infty$, i.e. the case where the Laspeyres aggregate is a consistent aggregate.

2.2 Paasche price indices

Having defined the Laspeyres import share indeces M^A and M^B, it remains to define the Paasche price index:

$$q(p^A,p^B) = \frac{c(p^A,p^B)}{m^A(p^A,p^B) + m^B(p^A,p^B)} = \frac{p^A m^A + p^B m^B}{m^A + m^B} =$$

$$= p^A M^A + p^B M^B = M^A (p^A - p^B) + p^B , \qquad (2.2.1)$$

where m^A, m^B M^A, and M^B are functions of the prices (p^A,p^B). Note that

$$M^A(p^A,p^B) = \Theta(p^A,p^B)\, m^A(p^A,p^B) ,$$

$$M^B(p^A,p^B) = \Theta(p^A,p^B)\, m^B(p^A,p^B) , \qquad (2.2.2)$$

$$q(p^A,p^B) = \Theta(p^A,p^B)\, c(p^A,p^B) ,$$

where

$$\Theta(p^A,p^B) = [m^A(p^A,p^B) + m^B(p^A,p^B)]^{-1} .$$

The Paasche aggregate $q(p^A,p^B)$ is <u>linearly homogeneous</u> in prices, and is defined for all positive prices for which $c(p^A,p^B)$ is defined and monotone increasing in at least one of its arguments (insuring that $m^A+m^B > 0$). We know [see f.ex. (2.1.3)] that $m^A+m^B \geq 1$. Thus it follows that

$$q(p^A,p^B) \leq c(p^A,p^B) . \qquad (2.2.3)$$

We are interested in studying in greater detail the behaviour of the Paasche index $q(p^A,p^B)$: is it monotone, concave...? Let us first consider its first derivatives, which we may compute, using the fact that $M^A + M^B = 1$ and eqs. (A1.2) in the appendix:

$$q_A = \frac{\partial q(p^A,p^B)}{\partial p^A} = M^A - c_{AB}\,\Theta^2 \frac{c}{p^A}(p^A - p^B)$$

$$= M^A \left[1 - \sigma \frac{M^B}{p^A}(p^A - p^B) \right]$$

$$= M^A \left[1 - \sigma\left(1 - \frac{q}{p^A}\right) \right], \qquad (2.2.4)$$

$$q_B = \frac{\partial q(p^A,p^B)}{\partial p^B} = M^B + c_{AB}\,\theta^2 \frac{c}{p^B}(p^A - p^B)$$

$$= M^B \left[1 + \sigma \frac{M^A}{p^B}(p^A - p^B) \right]$$

$$= M^B \left[1 - \sigma\left(1 - \frac{q}{p^B}\right) \right], \qquad (2.2.5)$$

where the elasticity of substitution

$$\sigma = \sigma(p^A,p^B) = \frac{c_{AB}(p^A,p^B)\; c(p^A,p^B)}{m^A(p^A,p^B)\; m^B(p^A,p^B)} \qquad (2.2.6)$$

depends on the prices. We have chosen to present three alternative expressions for q_A and q_B above.

It may be noted that Shephard's lemma does not hold for the Paasche index $q(p^A,p^B)$. Expect for the base point, where $p^A = p^B$, and $q_A = M^A = m^A$, and $q_B = M^B = m^B$, we have that

$$q_A \begin{cases} \leq M_A & \text{for } p^A > p^B , \\ \geq M_A & \text{for } p^A < p^B , \end{cases}$$

$$q_B \begin{cases} \geq M_B & \text{for } p^A > p^B , \\ \leq M_B & \text{for } p^A < p^B . \end{cases} \tag{2.2.7}$$

Additionally, while M_A and M_B are always non-negative, the derivatives of q may in fact be negative. We see from (2.2.4) that

$$q_A = 0 \qquad \text{if} \quad \frac{p^A}{p^B} = \frac{\sigma M^B}{\sigma M^B - 1} \quad (>1) \quad , \tag{2.2.8}$$

and from (2.2.5) that

$$q_B = 0 \qquad \text{if} \quad \frac{p^A}{p^B} = \frac{\sigma M^A - 1}{\sigma M^A} \quad (<1) \quad , \tag{2.2.9}$$

where M^A, M^B and σ are functions of the prices. The equations (2.2.8) and (2.2.9) may have multiple solutions. Let

$$\overline{\left(\frac{p^A}{p^B}\right)} = \inf \left\{ \frac{p^A}{p^B} \mid q_A(p^A,p^B) < 0 \right\} > 1 \ ,$$

$$\underline{\left(\frac{p^A}{p^B}\right)} = \sup \left\{ \frac{p^A}{p^B} \mid q_B(p^A,p^B) < 0 \right\} < 1 \ , \tag{2.2.9a}$$

and define the convex cone[4)]

---

4) Either or both of the sets defined in (2.2.9a) may be empty, in which case inf. = + ∞, and sup = − ∞.

$$\Pi \quad = \quad \{\ (p_A, p_B)\ |\ \underline{\left(\frac{p^A}{p^B}\right)} \leq \frac{p^A}{p^B} \leq \overline{\left(\frac{p^A}{p^B}\right)}\ ,\ p_A, p_B \geq 0\}\ . \tag{2.2.10}$$

Π is thus the largest convex cone containing the ray $[p^A = p^B]$ on which $q(p^A,p^B)$ is monotone nondecreasing in both its arguments. It follows also from (2.2.4) that

$$\begin{aligned} \frac{q}{p^A} &\geq 1 - \frac{1}{\sigma}\ , \\ \frac{q}{p^B} &\geq 1 - \frac{1}{\sigma}\ , \end{aligned} \tag{2.2.11}$$

for $(p^A,p^B) \in \Pi$.

The second derivative of $q(p^A,p^B)$ is

$$\begin{aligned} q_{AB}(p^A,p^B) &= \frac{dq_A}{dp^B} = \\ &= \sigma \frac{M^A M^B}{p^A p^B} [(p^A+p^B) + \sigma(M^A - M^B)(p^A - p^B)] - \\ &\qquad - M^A M^B \frac{p^A - p^B}{p^A} \frac{\partial \sigma}{\partial p^B} = \\ &= \sigma \frac{M^A M^B}{p^A p^B} \left[(1-\sigma)(p^A+p^B) + 2\sigma q - (p^A-p^B)\frac{p^B}{\sigma} \frac{\partial \sigma}{\partial p^B} \right] . \end{aligned} \tag{2.2.12}$$

Thus the second derivatives of q involve the derivatives of the elasticity of substitution, i.e. the third derivatives of the cost function. It seems difficult to have any opinion about the behaviour of this third derivative, but we note that this term will vanish at the base point, where $p_A = p_B = 1$. At the base point, with $M_A = m_A$, and $M_B = m_B$ we have in fact that:

$$q_{AB} \;=\; 2\,\sigma\, m^A m^B \;=\; 2\, c_{AB} \;\geq\; 0. \qquad (2.2.13)$$

Thus q is concave at the base point, and the implicit elasticity of substitution of the Paasche aggregate is twice that of the consistent aggregate.

The Paasche aggregate need not be concave. A counterexample will be provided below by the CES technology with $\sigma > 1$. We may however, by analogy with (2.2.10), define the convex cone Γ

$$\Gamma \;=\; \left\{ (p^A,p^B) \mid q(p^A,p^B) \text{ is concave, } [p^A=p^B] \in \Gamma \right\}, \qquad (2.2.14)$$

as the largest convex cone containing $[p^A=p^B]$ on which $q(p^A,p^B)$ is concave, i.e $q_{AB} \geq 0$. The cone Γ will be closed. It will in general not be possible to say anything specific about the relationship between the cones Π and Γ expect for the fact that their intersection will contain the ray $[p^A = p^B]$. But the properties of monotonicity and concavity seem to be of so great importance for a price index that we will define the neoclassical region

$$N \;=\; \Pi \cap \Gamma \qquad (2.2.15)$$

as the closed convex cone on which $q(p^A,p^B)$ behaves "as one would expect it should".

We may however show that the epigraph of q[5], and its level sets, have a more modest convex-like properly. A set S is said to be star shaped with respect to the point $x \varepsilon S$ if the line segment joining x to any other point in S is contained in S.

---

5) See Rockafeller (1970, p.23) for the definition of epigraph of a convex function. It is defined by

$$\text{epi}(q) \;=\; \{(p^A,p^B,q) \mid q \leq q(p^A,p^B)\}\;.$$

Lemma: the epigraph of q, epi(q), is star shaped with respect to the points on the ray $\{(p^A, p^B) \mid p^A = p^B\}$.

Proof: The proof will be more general than needed, in order to demonstrate an additional property of q. Define the convex combinations

$$p_0^A \quad = \quad \lambda p_1^A + (1-\lambda)\ {}_2^A \quad ,$$

$$p_0^B \quad = \quad \lambda p_1^B + (1-\lambda) p_2^B \quad .$$

Assume without loss of generality that $p_2^A/p_2^B \leq p_1^A/p_1^B$. This implies that

$$\frac{p_2^A}{p_2^B} \quad \leq \quad \frac{p_0^A}{p_0^B} \quad \leq \quad \frac{p_1^A}{p_1^B} \quad .$$

It follows from the monotonicity of M^B [see(2.1.5)] that

$$M^B(p_2) \quad \leq \quad M^B(p_0) \quad \leq \quad M^B(p_1) \quad . \tag{2.2.16}$$

We have used $p = (p^A, p^B)$ to designate the two-dimensional price vector. The line segment $[p_1, p_2]$ is contained in epi(q) if

$$q(p_0) \quad \geq \quad \lambda q(p_1) + (1-\lambda) q(p_2) \quad .$$

Using $q(p) = M^A(p)(p^A - p^B) + p^B$, and $M^A + M^B = 1$, gives

$$q(p_0) \quad = \quad \lambda\, M^A(p_0)(p_1^A - p_1^B) \quad + \quad (1-\lambda)\, M^A(p_0)\ (p_2^A - p_2^B) \quad + \quad p_0^B \quad ,$$

$$\lambda q(p_1) \quad + \quad (1-\lambda) q(p_2) \quad = \quad \lambda \; M^A(p_1) \; (p_1^A - p_1^B) \quad + \quad (1-\lambda) \; M^A(p_2)$$

$$(p_2^A - p_2^B) + p_0^B \quad ,$$

and

$$q(p_0) \quad - \quad \lambda \; q(p_1) \quad - \quad (1-\lambda) \; q(p_2) \quad =$$

(2.2.17)

$$= \quad \lambda \; [M^A(p_0) \; - M^A(p_1)] \; (p_1^A - p_1^B) \quad + \quad (1-\lambda) \; [M^A(p_2) \; - \; M^A(p_0)](p_2^B - p_2^A) \quad .$$

Both expressions inside the square brackets are positive by (2.2.16). Assume that p_1 belongs to the closed upper half of the positive orthant and p_2 to the closed lower half, i.e.

$$\frac{p_1^B}{p_1^B} \geq 1 \quad , \qquad \frac{p_2^A}{p_2^B} \leq 1 \quad ,$$

then $p_1^A - p_1^B \geq 0$, and $p_2^B - p_2^A \geq 0$ and the difference (2.2.17) is positive. This implies that any line segment $[p_1, p_2]$ connecting the two closed halfs of R_+^2 is contained in epi(q). In particular it implies that any line segment starting at a point on $[p^A = p^B]$ is contained in epi(q) since $[p^A = p^B]$ is contained in both closed halfs of R_+^2.

Let us consider the level sets of q. They will all be similar since q is linearly homogeneous, and we need therefore only consider the unit level set

$$L = \{ (p^A,p^B) \mid q(p^A,p^B) \geq 1 \} .$$

We know that (1,1) belongs to the boundary of L and that any ray from the origin intersects the boundary only once. We can further show that any ray formed by a price vector with strictly positive coordinates must intersect L.

Lemma: The translate $\{(1,1)\} + \bar{R}^2_+$ of the closed positive orthant belongs to L, i.e. $\{(1,1)\} + \bar{R}^2_+ \subset L$.

Proof: From (2.2.1) we know that the boundary of L is determined by the condition

$$p^B = 1 - M^A(p^A,p^B)\ (p^A-p^B) .$$

Assume that $p^A > p^B$. The expression on the left is bounded above by 1 since $M^A(p^A,p^B) \geq 0$, and we obtain that:

$$p^B \leq 1 \quad \text{on boundary of L for } p^A > p^B .$$

Similarly we can show that

$$p^A \leq 1 \quad \text{on boundary of L for } p^B < p^A .$$

The condition cannot be strengthened, since equality will be obtained if one of the inputs is non-essential. This will happen if for a sufficiently high p^A/p^B, m^A and thus M^A become zero.

2.3. CES technology

We will now analyze in greater detail the behaviour of the import share functions and the Paasche price index when the true technology can be represented by a CES function. The CES unit cost function can be written ($\sigma \neq 1$):

$$c(p^A,p^B) = [(1-\delta)(p^A)^{1-\sigma} + \delta(p^B)^{1-\sigma}]^{\frac{1}{1-\sigma}} , \qquad (2.3.1)$$

while the domestic and import coefficients are

$$m^A = (1-\delta)\left(\frac{p^A}{c}\right)^{-\sigma} , \qquad m^B = \delta\left(\frac{p^B}{c}\right)^{-\sigma} . \qquad (2.3.2)$$

Define the function

$$\lambda(p^A,p^B) = [(1-\delta)(p^A)^{-\sigma} + \delta(p^B)^{-\sigma}] . \qquad (2.3.3)$$

Then the sum of the coefficients becomes

$$\Theta^{-1} = m^A + m^B = \lambda\, c^{\sigma} . \qquad (2.3.4)$$

The Laspeyres domestic and import shares can be written:

$$M^A = (1-\delta)\,\frac{(p^A)^{-\sigma}}{\lambda} , \qquad M^B = \delta\,\frac{(p^B)^{-\sigma}}{\lambda} , \qquad (2.3.5)$$

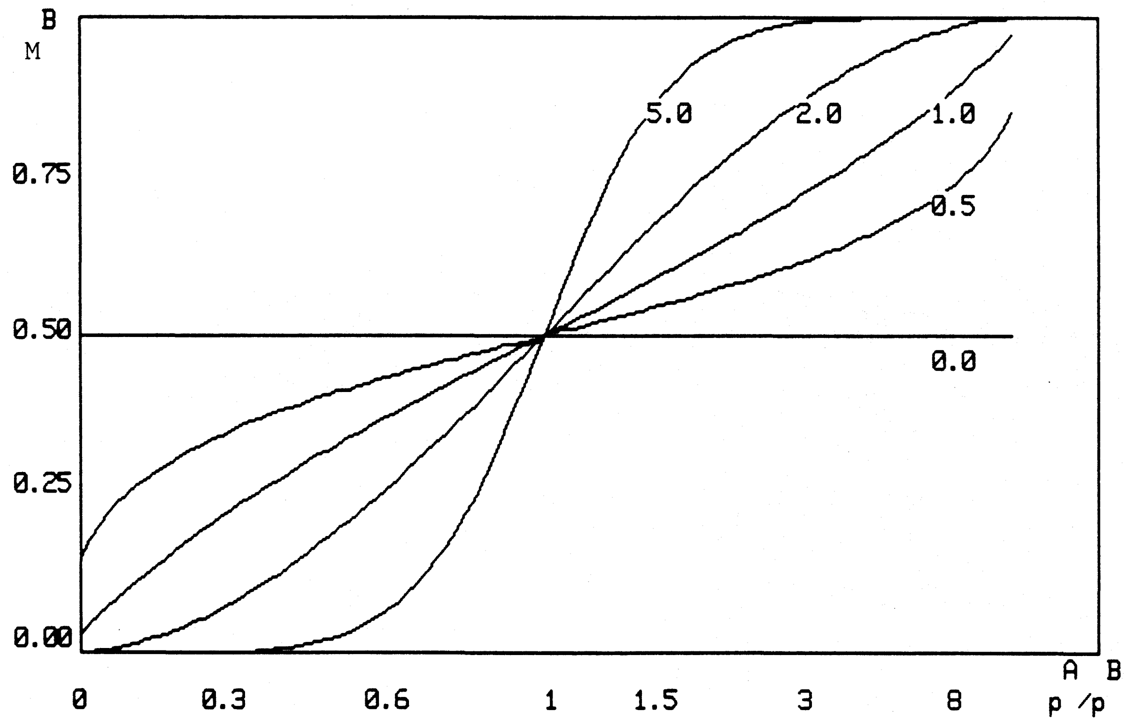
while the Paasche price index is

$$q = p^A M^A + p^B M^B = \lambda^{-1} c^{1-\sigma} =$$

$$= \frac{(1-\delta)\ (p^A)^{1-\sigma} + \delta\ (p^B)^{1-\sigma}}{(1-\delta)\ (p^A)^{-\sigma} + \delta\ (p^B)^{-\sigma}} \quad . \tag{2.3.6}$$

Figure 2.3.1 shows the import share as a function of relative prices p^A/p^B for five values of σ.

Fig. 2.3.1. - The import share M^B for various elasticities of substitution (δ=0,5)



Can the Paasche index be a decreasing function of prices when the true function is a CES function? We see directly from (2.2.4) or (2.2.11) that q_A, $q_B \geq 0$ when $\sigma \leq 1$. But, as we will now show, q_A and q_B will be negative for some relative prices if $\sigma>1$. Using (2.2.4), (2.3.5), and (2.3.6) the derivative of q with respect to the domestic price becomes:

$$q_A = (1-\delta)\frac{(p^A)^{-\sigma}}{\lambda}\left[1-\sigma\left(1-\frac{c^{1-\sigma}}{\lambda p^A}\right)\right] = \tag{2.3.7}$$

$$= (1-\delta)\lambda^{-2}(p^A)^{-\sigma-1}(p^B)^{1-\sigma}\left[(1-\delta)\left(\frac{p^A}{p^B}\right)^{1-\sigma} + (1-\sigma)\delta\left(\frac{p^A}{p^B}\right) + \sigma\delta\right] .$$

Let t represent the relative price ($t=p^A/p^B$). Then $q_A = 0$ when

$$\phi(t) = (1-\delta)t^{1-\sigma} + (1-\sigma)\delta t + \sigma\delta \tag{2.3.8}$$

equals zero for som $t>0$. It is not possible to solve (2.3.8) explicitly, but $\phi(1) = 1$ and $\phi(t)$ will be negative for a sufficiently large t. Thus $\phi(t)$, being continuous, will be zero somewhere in between. Further note that

$$\phi'(t) = (1-\sigma)\ [(1-\delta)t^{-\sigma} + \delta] \quad ,$$

which is negative for $\sigma>1$. Thus $\phi(t)$ is a decreasing function on $t\varepsilon(0,\infty)$ and q_A can be zero for at most one relative price.

The second derivatives of the Paasche index are readily obtained using (2.2.12). By the very definition of a CES function $\partial\sigma/\partial p^B = 0$, and

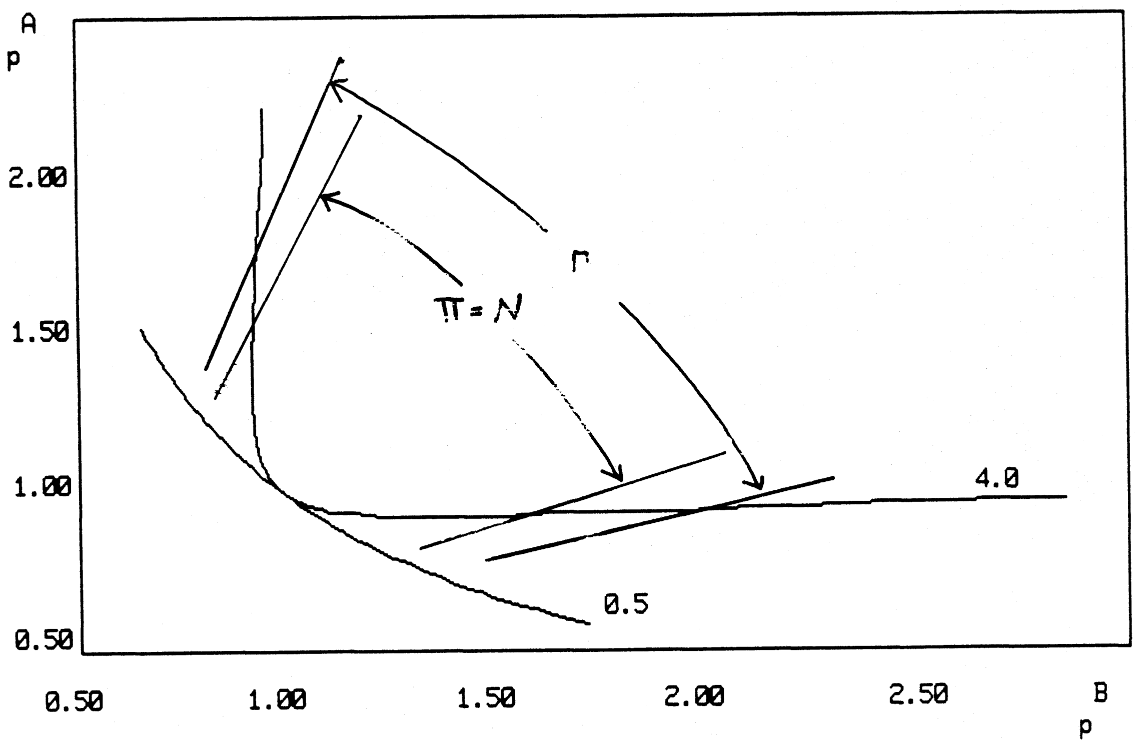
$$q_{AB} = \sigma\frac{M^A M^B}{p^A p^B}[(1-\sigma)(p^A+p^B) + 2\sigma q] \quad . \tag{2.3.9}$$

It follows that $q_{AB} \geq 0$ and q is concave for $\sigma \leq 1$. But the Paasche price index will not be concave for $\sigma > 1$. Using (2.2.11) we can however show that for $(p^A, p^B) \in \Pi$, the set on which q_A and q_B are nondecreasing[1),

$$q_{AB} \;=\; \sigma^2 \frac{M^A M^B}{p^A p^B} \{(q + (1-\frac{1}{\sigma})p^A) \;+\; (q + (1-\frac{1}{\sigma})p^B)\} \;\geq\; 0 \quad . \text{(2.3.10)}$$

This shows that the cone Γ on which q is concave [see (2.2.14)] contains Π and that the neoclassical region N [see (2.2.15)] coincides with Π (i.e $N=\Pi$). We have thus shown that the Paasche price index behaves neoclassically for all prices if $\sigma \leq 1$, but that it is no longer neither monotone nor concave if $\sigma > 1$. These conclusions are illustrated in fig. 2.3.2 where we present the "isocosts" (or level curves) for the Paasche price index for two elasticities of substitution: $\sigma = 0.5$ and $\sigma = 4$.

Fig. 2.3.2 - Isocost curves for the Paasche price index ($\delta = 0.5$)



1) See (2.2.10) above for definition of Π.

We see readily that the isocost is concave for all prices when $\sigma = 0{,}5$, while it is only concave for relative prices in the cone Γ for $\sigma = 4$.

3. A neoclassical import model

In the previous section we analyzed the effects of using inconsistent Laspeyres and Paasche aggregates in describing the input of a single commodity into a single sector. In the next section we will incorporate these inconsistent indeces in a simultaneous input-output model. Before doing that, however, it will be convenient to summarize the behaviour of the consistently aggregated input-output model.

Let a_{ik} represent the input coefficient of commodity i, defined as a function [see (1.1)] of the imported and domestically produced input, into sector k, and let a_{Vk} represent the input coefficient for value added. These coefficients will be assumed constant, and may for example have been estimated from the base year national accounts. Further let p_k^V be a price index for value added in sector k. The simultaneous neoclassical price model[1] may now be written as:

$$\begin{aligned} p_k^A &= \sum_i a_{ik}\, c^{ik}(p_i^A, p_i^B) + a_{Vk} p_k^V = & k=1,\dots,m \\ &= \sum_i a_{ik}\, [p_i^A\, m_{ik}^A(p_i^A, p_i^B) + p_i^B\, m_{ik}^B(p_i^A, p_i^B)] + a_{Vk} p_k^V \quad , & (3.1) \end{aligned}$$

The cost functions c^{ik} are normalized so that $c^{ik}(1,1) = 1$. The model (3.1) defines implicitly the domestic prices p^A as functions of the prices of imported commodities p^B and the price indeces for value added:

$$p^A = p(p^B, p^V) \quad . \tag{3.2}$$

1) A more thorough analysis of such a model is presented in Frenger (1979b)

The functions $p_k^A = p^k(p^B,p^V)$, k=1,...,m, represent the total direct and indirect costs of producing a single unit of the i'th commodity. They are in fact true cost functions, and are linear homogeneous and concave in the prices (p^B,p^V). Differentiating (3.1) totally, and solving with respect to the endogenous price changes gives , using matrix notation , that

$$dp^A = [I - (m^A oA)']^{-1}[(m^B oA)'dp^B + a_V' \, dp^V] \quad , \tag{3.3}$$

where m^A and m^B depend on the prices p^A and p^B. The expression shows that

$$J_p = \frac{dp^A}{dp^B} = [I - (m^A oA)']^{-1} (m^B oA)' \geq 0, \tag{3.4}$$

represents the matrix of first derivatives of p^A with respect to p^B evaluated at $(p^A(p^B,p^V),p^B,p^V)$. The matrix J_p represents also, for given p^V, the Jacobian of the transformation from p^B to p^A. It follows that if (3.1) has a non-negative solution, then

$$[\hat{p}^{A^{-1}} (m^A oA)' \, \hat{p}^A] \, e < e \quad , \tag{3.5}$$

where e is a vector of ones. This insures also that

$$I - (m^A oA)' = \hat{p}^A[I - \hat{p}^{A^{-1}} (m^A oA)'\hat{p}^A]\hat{p}^{A^{-1}} \tag{3.6}$$

satisfies the Hawkins - Simon condition[2], and that $J_p \geq 0$, i.e. the elements of J_p are non-negative.

---

2) The matrices $[I - (m^A oA)']$ and $[I - \hat{p}^{A^{-1}} (m^A oA)'\hat{p}^A]$ are similar, and thus have the same set of eigenvalues to which 1 is an upperbound.

Using our interpretation of $p(p^B,p^V)$ as unit cost functions and applying Shephard's lemma, shows that we may interpret J_p' as an import share matrix where element (i,j) represents the total direct and indirect import requirement of commodity i necessary to produce a single unit of commodity j.

An explicit derivation of the second derivatives of $p(p^B,p^V)$ is presented in Frenger (1979b, pp. 35-6). The computations are tedious in part because these second derivatives represent a three dimensional matrix, and I will therefore only present the conclusions. Define the matrix of elasticities of substitutions

$$\Sigma \quad = \quad [\sigma_{ik}] \quad = \quad \left[\frac{\partial c^{ik}}{\partial p_i^A \partial p_i^B} \, c^{ik} \Big/ m_{ik}^A m_{ik}^B \right] \quad , \tag{3.7}$$

and the second order matrix

$$D \quad = \quad (\Sigma \circ m^A \circ m^B \circ A) \, [I - (m^A \circ A)]^{-1} \quad . \tag{3.8}$$

Further let $\hat{D}^k$ represent the k'th column of D written as a diagonal matrix. The matrix of second derivatives of the domestic price p_k^A with respect to the import price vector p^B can then be written

$$H_p^k \quad = \quad \frac{d^2 p_k^A}{(dp^B)^2} \quad = \quad - (J_p' - I) \, \hat{D}^k \, (J_p - I) \quad . \tag{3.9}$$

All the elements of D^k are non-negative, and it follows that H_p^k is negative semidefinite, as one would expect from the concavity of $p^k(p^B,p^V)$.

4. A Paasche import model

We will now return to the inconsistently aggregated domestic and import share indeces M_{ik}^{A} and M_{ik}^{B} [see (2.1.1) and (2.1.2)], and the Paasche price index q_{ik} [see (2.2.1)] presented in section 2, and incorporate them in the input-output model described in the previous section. I will remind the reader that the input of the i'th commodity into the k'th sector x_{ik} is given as a composite of the domestically produced input x_{ik}^{A} and the imported input x_{ik}^{B}, and that the relationship is described by the neoclassical production function [see (1.1)]

$$x_{ik} = f^{ik}(x_{ik}^{A}, x_{ik}^{B}) \quad . \tag{4.1}$$

The function f^{ik} is in general unknown and is approximated by the Laspeyres quantity index

$$x_{ik} = x_{ik}^{A} + x_{ik}^{B} \quad .$$

This defines implicitly the Paasche price index

$$q_{ik} = \frac{p_i^A x_{ik}^A + p_i^B x_{ik}^B}{x_{ik}^A + x_{ik}^B} = \frac{c^{ik}(p_i^A, p_i^B)}{m_{ik}^A(p_i^A, p_i^B) + m_{ik}^B(p_i^A, p_i^B)} \quad , \tag{4.2}$$

where I, in the second equality, have allowed for the fact that the sector determines its factor demand via cost minimization and a knowledge of the true technology (4.1). We have previously shown that the indeces satisfy the inequalities [see (2.1.3) and (2.2.3)]:

$$m_{ik}^{A}(p_i^A,p_i^B) \;+\; m_{ik}^{B}(p_i^A,p_i^B) \;\geq\; 1 \;,$$

$$q^{ik}(p_i^A,p_i^B) \;\leq\; c^{ik}(p_i^A,p_i^B) \;. \tag{4.3}$$

Let us now use the Paasche price indeces (4.2) to determine the prices in the input-output model of section 3. This gives us m price functions

$$q_k \;=\; \sum_i a_{ik}\, q^{ik}(p_i^A,p_i^B) \;+\; a_{Vk}\, p_k^V \;, \qquad k=1,\ldots,m. \tag{4.4}$$

Making the obvious notational identifications this may be rewritten

$$q \;=\; Q(p^A,p^B,p^V) \;, \tag{4.5}$$

which represents a transformation from R^{3m} to R^m.

It should be emphasized that the model (4.4) or (4.5) is not the model of the behaviour of the production sectors of the economy. Their behaviour is still described by the neoclassical model of sector 3. The relationship (4.4) or (4.5) represent the model builder's erroneous representation of the sectors which results from his use of inconsistent aggregates.

The model (4.4) does not determine the domestic prices p^A. We will however assume that the model builder "closes" his model in the usual way by setting price equal to unit cost, i.e. by setting p^A=q. Adding this condition to (4.4) determines the <u>Paasche price model</u>

$$q_k \;=\; \sum_i a_{ik}\, q^{ik}(q_i,p_i^B) \;+\; a_{Vk}\, p_k^V \;, \qquad k=1,\ldots,m, \tag{4.6}$$

which may be compared with the neoclassical model (3.1). The Paasche model determines implicitly q as a function of p^B and p^V [1)]

$$q \quad = \quad q(p^B, p^V) \quad . \tag{4.7}$$

We will now analyze the behaviour of this model more closely. How does the Paasche model differ from the neoclassical model? Can (4.7) be interpreted as a set of cost functions whose first derivatives determine the import shares? In answering these, and other, questions it will be important to remember that we are always assuming that the true underlying model is described by a neoclassical technology and that the producer minimizes cost. The results will be of two types: we will first present local results about the base point and then give a global result. [2)]

i) <u>local results</u>

First of all it follows from the normalization of the indeces that (4.7) satisfies the base period (or base point) identity

$$q(1,1) \quad = \quad 1 \quad . \tag{4.8}$$

Differentiating totally the Paasche price model (4.6) and using the identity $q_{ik} = \theta_{ik}c^{ik}$ [see (2.2.2)] gives

---

1) This solution is a fixed point of the mapping $Q(\cdot,p^B,p^V)$ from R^m to R^m.

2) The two sets of conclusions are derived independently. The global results, being based on concavity, are the easiest to follow.

$$dq_k = \sum_i a_{ik}\, \theta_{ik} \left(\frac{\partial c^{ik}}{\partial q_i} dq_i + \frac{\partial c^{ik}}{\partial p_i^B} dp_i^B\right)$$

$$+ \sum_i a_{ik}\, c^{ik} \left(\frac{\partial \theta_{ik}}{\partial q_i} dq_i + \frac{\partial \theta_{ik}}{\partial p_i^B} dp_i^B\right) + a_{Vk}\, dp_k^V$$

$$= \sum_i a_{ik}\, \theta_{ik}\, (m_{ik}^A\, dq_i + m_{ik}^B\, dp_i^B)$$

$$- \sum_i a_{ik}\, c_{AB}^{ik}\, c^{ik}\, \theta_{ik}^2\, (q_i - p_i^B)\left(\frac{dq_i}{q_i} - \frac{dp_i^B}{p_i^B}\right) + a_{Vk}\, dp_k^V \; .$$

We have also used the homogeneity of θ_{ik} and (A.1.1).[3] Define the matrix

$$R = [a_{ik}\, c_{AB}^{ik}\, c^{ik}\, \theta_{ik}^2] = [a_{ik}\, \sigma_{ik}\, M_{ik}^A\, M_{ik}^B] \; . \tag{4.9}$$

and rewrite the above expression in matrix form

$$dq = (m^A \circ \Theta \circ A)'\, dq + (m^B \circ \Theta \circ A)'\, dp^B -$$

$$- R'(\hat{q} - \hat{p}^B)(\hat{q}^{-1} dq - \hat{p}^{B^{-1}} dp^B) + \hat{a}_V\, dp^V \; .$$

Solving this expression for dq gives the first derivatives of the Paasche price model with respect to the import prices

---

3) The first derivatives of θ_{ik} are derived in appendix 1.

$$J_q = \frac{dq}{dp^B} = [I - (M^A oA)' + R'(\hat{q}-\hat{p}^B)\hat{q}^{-1}]^{-1} \; [(M^B oA)' +$$

$$+ R'(\hat{q}-\hat{p}^B)\hat{p}^{B^{-1}}] \qquad (4.11)$$

estimated at the point $q(p^B, p^V)$, p^B, p^V . One should remember that Θ , m^A, m^B, and R are all functions of q, p^B, and p^V .

It follows that the Jacobian (4.11) reduces to

$$\frac{dq}{dp^B} = [I - (m^A oA)']^{-1} \; (m^B oA)' \qquad (4.12)$$

at the base point where $q_i = p_i^A = p_i^B = 1$. This expression is the same as the Jacobian (3.4) of the neoclassical model, and the Paasche model may therefore be interpreted as a first order approximation to the true neoclassical model. It follows from the inequality in (3.4) that $dq/dp^B \geq 0$, but the results of section 2 indicate that this inequality need not hold at other points.

The second derivatives of $q(p^B, p^V)$ will depend upon the third derivatives of the technology. These third order terms vanish, however, at the base point, and it is shown in appendix 1 that at this point the Hessian of $q^k(p^B, p^V)$ with respect to p^B may be written [see (A1.11)]

$$H_q^k = -2 \; (J_q - I)' \; \hat{D}^k \; (J_q - I) \quad . \qquad (4.13)$$

D^k is defined in section 3, and H_q^k may be compared with H_p^k [see (3.9)] . We see that at the base point

$$H_q^k = 2 \; H_p^k \quad , \qquad (4.14)$$

a result which generalizes (2.2.13). Thus H_q^k is negative semidefinite and "locally concave".

Summarizing the results of this section thus far, we see that the Paasche price model $q(p^B,p^V)$ provides a first order approximation to the neoclassical model $p(p^B,p^V)$ at the base point (1,1). It follows from (4.14) that there is a neigbourhood N_ε of (1,1) such that

$$q(p^B,p^V) \leq p(p^B,p^V) \quad \text{on} \quad N_\varepsilon \ . \tag{4.15}$$

We showed in section 2 that this inequality had to hold for all import prices in the simple non-simultaneous model. Does a similar conclusion hold in the simultaneous case outside N_ε , i.e. can we show that (4.15) holds for all positive prices (p^B,p^V) ?

ii) global results

Let us rewrite the neoclassical price equations (3.1) along the lines of equation (4.5)

$$p^C = P(p^A,p^B,p^V) \quad , \tag{4.16}$$

where we have defined the "cost" price p^C . The neoclassical price model (3.2) is just a fixed point of $P(\cdot,p^B,p^V)$. It follows from the second inequality in (4.3) that

$$Q(p^A,p^B,p^V) \leq P(p^A,p^B,p^V) \quad , \tag{4.17}$$

when these functions are evaluated at the same point in R^{3m}. But this is not sufficient to show that (4.15) holds for all (p^B, p^V) since $Q(\cdot, p^B, p^V)$ and $P(\cdot, p^B, p^V)$ will in general not have the same fixed point. The following argument will, however, show that (4.15) holds for all import prices.

Assume that q is a solution to the Paasche model i.e. that

$$q = q(p^B, p^V) = Q(q(p^B, p^V), p^B, p^V) .$$

We see from (4.17) that

$$q = Q(q, p^B, p^V) \leq P(q, p^B, p^V) .$$

Let p^A be a solution of $p^A = P(p^A, p^B, p^V)$, then it follows from theorem 4.1 (see appendix 2) that

$$q \leq p^A , \tag{4.18}$$

and thus that the endogenously determined prices in the Paasche model all are lower than the corresponding prices derived from the neoclassical model.

5. Conclusions

In concluding it may be worth-while to summarize the consequences of using inconsistent Laspeyres and Paasche indeces in a neoclassical (import) model. Section 2 showed that we would tend to underestimate both the factor demand and the output price index. This is a simple consequence of the inequalities (2.1.4) and (2.2.3). Let us call this the index bias. This bias was illustrated graphically in figure 2.1.1 by the distance between the points B (the neoclassical solution) and the point C (the inconsistent solution).

In section 2 we assumed that the true price p^A of the domestic inputs was used to determine factor demand and output prices. But when the inconsistent indeces are incorporated in a simultaneous model we get an additional bias represented by the inequality $q \leq p^A$ [see (4.18)] . Thus all inconsistent price ratios between domestically produced and imported inputs are lower than the true ratios

$$\frac{q_i}{p_i^B} \leq \frac{p_i^A}{p_i^B}, \qquad i=1,\dots,m.$$

Let us call this the model bias. It will tend to overestimate the demand for domestic inputs (which appear cheaper) and underestimate the demand for imported inputs. Returning to figure 2.1.1 we see that the model bias and the increase in the price of imports would have given us a relative price line P_1-P_1 which was even flatter than the one shown in the figure.

The combined effect of the index bias and the model bias will leave the net effect on the demand for domestic inputs uncertain, the two effects having opposite sign, but it will underestimate the demand for imports and the level of the domestic prices.[1)]

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1) Remember, however, that $q=q(p^B,p^V)$ is linearly homogeneous in (p^B,p^V) .

Appendix 1 - Second derivatives of the Paasche model

This appendix will present the computations of the second derivatives of the Paasche model (4.6). We will start by computing the derivatives of the θ_{ik}, and the M^A_{ik} and M^B_{ik} functions [see (2.2.2), (2.1.1), (2.1.2)], which were already needed in section 2. We express θ_{ik} as a function of p^A_i and p^B_i :

$$\theta_{ik}(p^A_i, p^B_i) = [m^A_{ik}(p^A_i, p^B_i) + m^B_{ik}(p^A_i, p^B_i)]^{-1} .$$

Its derivatives are

$$\frac{\partial \theta_{ik}}{\partial p^A_i} = - c^{ik}_{AB}\, \theta^2_{ik} \frac{p^A_i - p^B_i}{p^A_i} ,$$

$$\frac{\partial \theta_{ik}}{\partial p^B_i} = c^{ik}_{AB}\, \theta^2_{ik} \frac{p^A_i - p^B_i}{p^B_i} = - \frac{p^A_i}{p^B_i} \frac{\partial \theta_{ik}}{\partial p^A_i} . \tag{A 1.1}$$

The expression c^{ik}_{AB} represents the second derivative of the unit cost function c^{ik} with respect to p^A_i and p^B_i . The second equality above is just a confirmation of the fact that θ_{ik} is homogeneous of degree zero in prices.

The first derivatives of the domestic and import share functions are

$$\frac{\partial M_{ik}}{\partial p^A_i} = \frac{\partial}{\partial p^A_i} (m^A_{ik} \theta_{ik}) = - c^{ik}_{AB}\, \theta^2_{ik} \frac{c^{ik}}{p^A_i} = - \sigma_{ik} \frac{M^A_{ik} M^B_{ik}}{p^A_i} ,$$

(A 1.2)

$$\frac{\partial M_{ik}^{A}}{\partial p_i^{B}} = \sigma_{ik} \frac{M_{ik}^{A} M_{ik}^{B}}{p_i^{B}} = - \frac{p_i^{A}}{p_i^{B}} \frac{\partial M_{ik}^{A}}{\partial p_i^{A}} .$$

$$\frac{\partial M_{ik}^{B}}{\partial p_i^{A}} = c_{AB}^{ik} \theta_{ik} \left(1 - M_{ik}^{B} \frac{p_i^{A} - p_i^{B}}{p_i^{A}}\right) = - \frac{\partial M_{ik}^{A}}{\partial p_i^{A}} ,$$

(A 1.3)

$$\frac{\partial M_{ik}^{B}}{\partial p_i^{B}} = - \frac{p_i^{A}}{p_i^{B}} \frac{\partial M_{ik}^{B}}{\partial p_i^{A}} = \frac{p_i^{A}}{p_i^{B}} \frac{\partial M_{ik}^{A}}{\partial p_i^{A}} .$$

The functions M_{ik}^{A} and M_{ik}^{B} are homogeneous of degree zero in prices and they sum to one. This makes it possible, as we see, to express the four derivatives as a function of any one of them.

In the simultaneous model, the neoclassical model of section 3 and the Paasche model of section 4, the domestic price (p^{A} or q) becomes itself a function of the import price p^{B}. The "total derivative" (i.e. taking into consideration the simultaneous effect), of the share functions can be written (δ_{ij}^{K} is the Kronecker delta)

$$\frac{dM_{ik}^{A}}{dp_j^{B}} = \frac{\partial M_{ik}^{A}}{\partial p_i^{A}} \frac{dp_i^{A}}{dp_j^{B}} + \frac{\partial M_{ik}^{A}}{\partial p_i^{B}} \frac{dp_i^{B}}{dp_j^{B}} =$$

$$= - c_{AB}^{ik} \theta_{ik}^{2} \frac{c^{ik}}{p_i^{B}} \left(\frac{dp_i^{A}}{dp_j^{B}} \frac{p_i^{B}}{p_i^{A}} - \delta_{ij}^{K}\right) ,$$

(A 1.4)

$$\frac{dM_{ik}^{B}}{dp_j^{B}} = - \frac{dM_{ik}^{A}}{dp_j^{B}} .$$

Thus far we have used the notation of the neoclassical model. When applying the results to the Paasche model one only has to substitute q for p^A. For the rest of this appendix we will discuss the Paasche model only and will therefore use q instead of p^A. Our main task will be the differentiation of the elements of the Jacobian matrix J_q [see (4.11)] with respect to the import prices p^B. Define the matrices [see (4.9) for the definition of R]

$$U' = I - (M^A \circ A)' + R' (\hat{q} - \hat{p}^B) \hat{q}^{-1} ,$$

$$V' = (M^B \circ A)' + R' (\hat{q} - \hat{p}^B) \hat{p}^{B^{-1}} ,$$

and rewrite the Jacobian as

$$J_q = U'^{-1} V' .$$

The derivatives of the Jacobian matrix J_q with respect to the import price p_j^B may be written

$$\frac{dJ_q}{dp_j^B} = - U'^{-1} \frac{dU'}{dp_j^B} U'^{-1} V' + U'^{-1} \frac{dV'}{dp_j^B}$$

$$= - U'^{-1} \left(\frac{dU'}{dp_j^B} J_q - \frac{dV'}{dp_j^B} \right) , \qquad j=1,\dots,m. \qquad \text{(A 1.5)}$$

The derivatives of the materices U' and V' in (A 1.5) are

$$\frac{dU_{ik}}{dp_j^B} = -a_{ik}\frac{dM_{ik}^A}{dp_j^B} + \frac{dr_{ik}}{dp_j^B}\frac{q_i-p_i^B}{q_i} + \frac{r_{ik}}{q_i}\left(\frac{dq_i}{dp_j^B}\frac{p_i^B}{q_i} - \delta_{ij}^K\right),$$

(A 1.6)

$$\frac{dV_{ik}}{dp_j^B} = a_{ik}\frac{dM_{ik}^B}{dp_j^B} + \frac{dr_{ik}}{dp_j^B}\frac{q_i-p_i^B}{p_i^B} + \frac{r_{ik}}{q_i}\left(\frac{q_i}{p_i^B}\right)^2\left(\frac{dq_i}{dp_j^B}\frac{p_i^B}{q_i} - \delta_{ij}^K\right).$$

We see that these derivatives involve the first derivatives of r_{ik} and thus third derivatives of the cost functions c^{ik}. The computations become rather complicated, and it seems difficult to form any definite opinion about these third derivatives. We will therefore restrict the computations to the base point, where the term with the third derivatives vanish since $q_i = p_i^B = 1$.[1] As $dM_{ik}^B/dp_j^B = -dM_{ik}^A/dp_j^B$ [see (A 1.4)], the derivatives of U' and V' are equal at the base point

$$\frac{dU_{ik}}{dp_j^B} = \frac{dV_{ik}}{dp_j^B} = (a_{ik}c_{AB}^{ik} + r_{ik})\left(\frac{dq_i}{dp_j^B} - \delta_{ij}^B\right)$$

$$= 2r_{ik}\left(\frac{dq_i}{dp_j^B} - \delta_{ij}^K\right), \qquad \text{(A 1.7)}$$

where $c^{ik} = \theta_{ik} = 1$. Let us define $(J_q-I)^j$ as a (column-)vector consisting of the elements of the j'th column of J_q-I. The derivatives (A 1.7) can then be written

$$\frac{dU}{dp_j^B} = \frac{dV}{dp_j^B} = 2\,\widehat{(J_q - I)^j R}\,, \qquad \text{(A 1.8)}$$

---

1) We could alternatively have supposed that these second derivatives were constant or that the elasticity of substitution was constant (CES technology). It is also possible that the magnitude of the third derivatives is irrelevant for the conclusions we wish to draw, particularly regarding the concavity of the price functions.

and the derivatives of the Jacobian matrix (A 1.5) become

$$\frac{dJ_q}{dp_j^B} = -U'^{-1}\frac{dU'}{dp_j^B}(J_q - I) = -2\,U'^{-1}R'\,\widehat{(J_q - I)^j}\,(J_q - I)\ . \tag{A 1.9}$$

The set dJ_q/dp_j^B, $j=1,\dots,m$, forms a three dimensional matrix. We are particularly interested in the matrix $H_q^k = [d^2q_k/(dp_i^B dp_j^B)]$, $i,j=1,\dots,m$, which represents the second derivatives of the domestic price index q_k with respect to the import prices. We see that the vector consisting of the second derivatives of q_k with respect to p^B and p_j^B forms the k'th row of dJ_q/dp_j^B:

$$\frac{d^2q_k}{dp^B dp_j^B} = -2\,(U'^{-1})_k\,R'\,\widehat{(J_q - I)^j}\,(J_q - I)$$

$$= -2\,(J'_q - I)_j\,\widehat{[(U'^{-1})_k R']}\,(J_q - I)\ , \tag{A 1.10}$$

where $(U'^{-1})_k$ is the k'th row of U'^{-1} and $(J'_q - I)_j$ is the j'th row of $(J'_q - I)$. The complete Hessian matrix H_q^k may now be written

$$H_q^k = \frac{d^2q}{(dp^B)^2} = \begin{bmatrix} \dfrac{d^2q_k}{dp^B dp_1^B} \\ \vdots \\ \dfrac{d^2q_k}{dp^B dp_m^B} \end{bmatrix}$$

$$= -2(J_q - I)'\,\widehat{[(U'^{-1})_k R']}\,(J_q - I)\ .$$

The elements of the diagonal matrix $\widehat{[(U'^{-1})_k R']}$ are given by the k'th column of the matrix

$$D = [d_{ik}] = \left[\sum_j r_{ij} \, [I - (m^A oA)]^{-1}_{jk} \right]$$

$$= R \, [I - (m^A oA)]^{-1} > 0 \ .$$

Let D^k represent the k'th column of D. Then the Hessian can be written

$$H^k_q = - 2 \, (J_q - I)' \, \widehat{D^k} \, (J_q - I) \ . \qquad \text{(A 1.11)}$$

This matrix is symmetric and negative semidefinite. The latter follows from the fact that the elements of D^k are non-negative.

Appendix 2 - Proof of theorem 4.1

This appendix will state and prove the theorem which we used in section 4 to show that the inequality $q \leq p^A$ [see (4.18)] holds for all import prices.

Theorem 4.1. Let $F = (f^1,\dots,f^m)$ be a continous mapping from R^m to R^m which is .

a) monotone nondecreasing in each argument

b) "sub-linearly" homogeneous, i.e.

$$f^k(\lambda x) < \lambda f^k(x) \quad , \qquad \lambda > 1 \qquad k=1,\dots,m.$$

Let x^0 be a fixed point, i.e. $x^0 = F(x^0)$, and let x^1 be another point such that $x^1 \leq F(x^1)$. Then $x^0 \geq x^1$.

Proof: Assume on the contrary that $x^0 \not\geq x^1$. Partition the index set $\{1,\dots,m\}$ into two sets S and T such that

$$x_k^0 < x_k^1 \quad . \quad k \varepsilon S \ ,$$

$$x_k^0 \geq x_k^1 \qquad k \varepsilon T \ ,$$

and define

$$\lambda = \max\{ \frac{x_k^1}{x_k^0} , k \varepsilon S \} > 1 \ .$$

Let $i \varepsilon S$ be an index for which $\lambda = x_i^1/x_i^0$, and assume, after reordering if necessary, that the indeces are so ordered that the argument of the function f^i say be written

$$f^i(x^1) = f^i(\overbrace{x_1^1, x_2^1, \dots x_s^1}^{S}, \overbrace{x_{s+1}^1, \dots , x_m^1}^{T}) =$$

$$= f^i(\lambda\lambda^{-1}x_S^1, \lambda\lambda^{-1}x_T^1) < \lambda f^i(\lambda^{-1}x_S^1, \lambda^{-1}x_T^1) .$$

From the definition of λ follows that $\lambda^{-1}x_k^1 \leq x_k^0$ for every $k \varepsilon S$. And since $\lambda^{-1} < 1$ it follows that $\lambda^{-1}x_k^1 < x_k^0$ for every $k \varepsilon T$. Thus

$$f^i(\lambda^{-1}x_S^1, \lambda^{-1}x_T^1) \leq f^i(x_S^0, x_T^0) = f^i(x^0) .$$

Hence

$$f^i(x^1) < \lambda f^i(x^0) = \lambda x_i^0 = x_i^1 .$$

But this contradicts the assumption that $x^1 \leq F(x^1)$.

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